Electrical and Computer Engineering Seminar Series

Flocking dynamics promoted by heterogeneity

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Center for Network Dynamics

Department of Physics and Astronomy

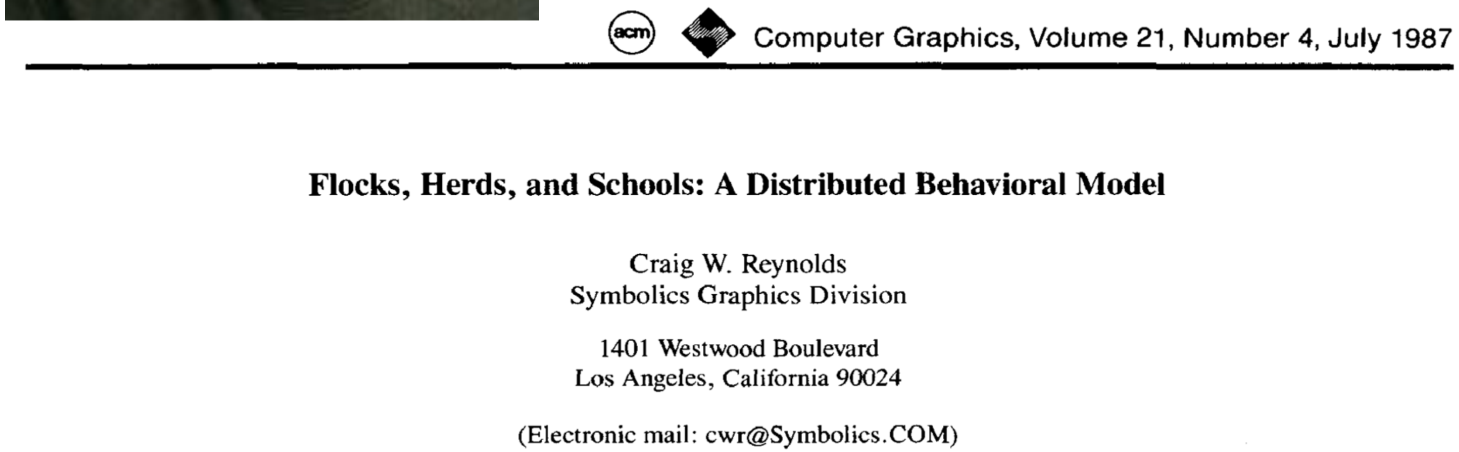
Northwestern University

December 4, 2024

NATIONAL GEOGRAPHIC

We know a lot of factual information about the starling—its size and voice, where it lives, how it breeds and migrates—but what remains a mystery is how it flies in murmurations, or flocks, without colliding.

Computer animations



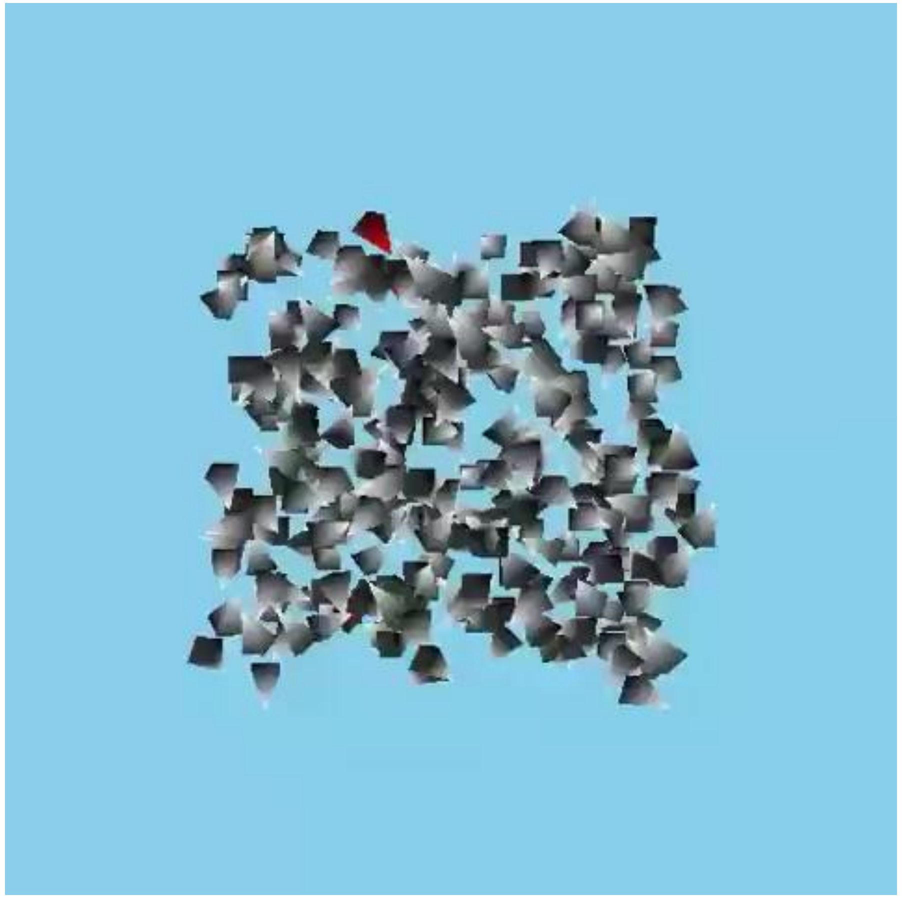
Computer Graphics, Volume 21, Number 4, July 1987

Flocks, Herds, and Schools: A Distributed Behavioral Model

Craig W. Reynolds
Symbolics Graphics Division

1401 Westwood Boulevard
Los Angeles, California 90024

(Electronic mail: cwr@Symbolics.COM)

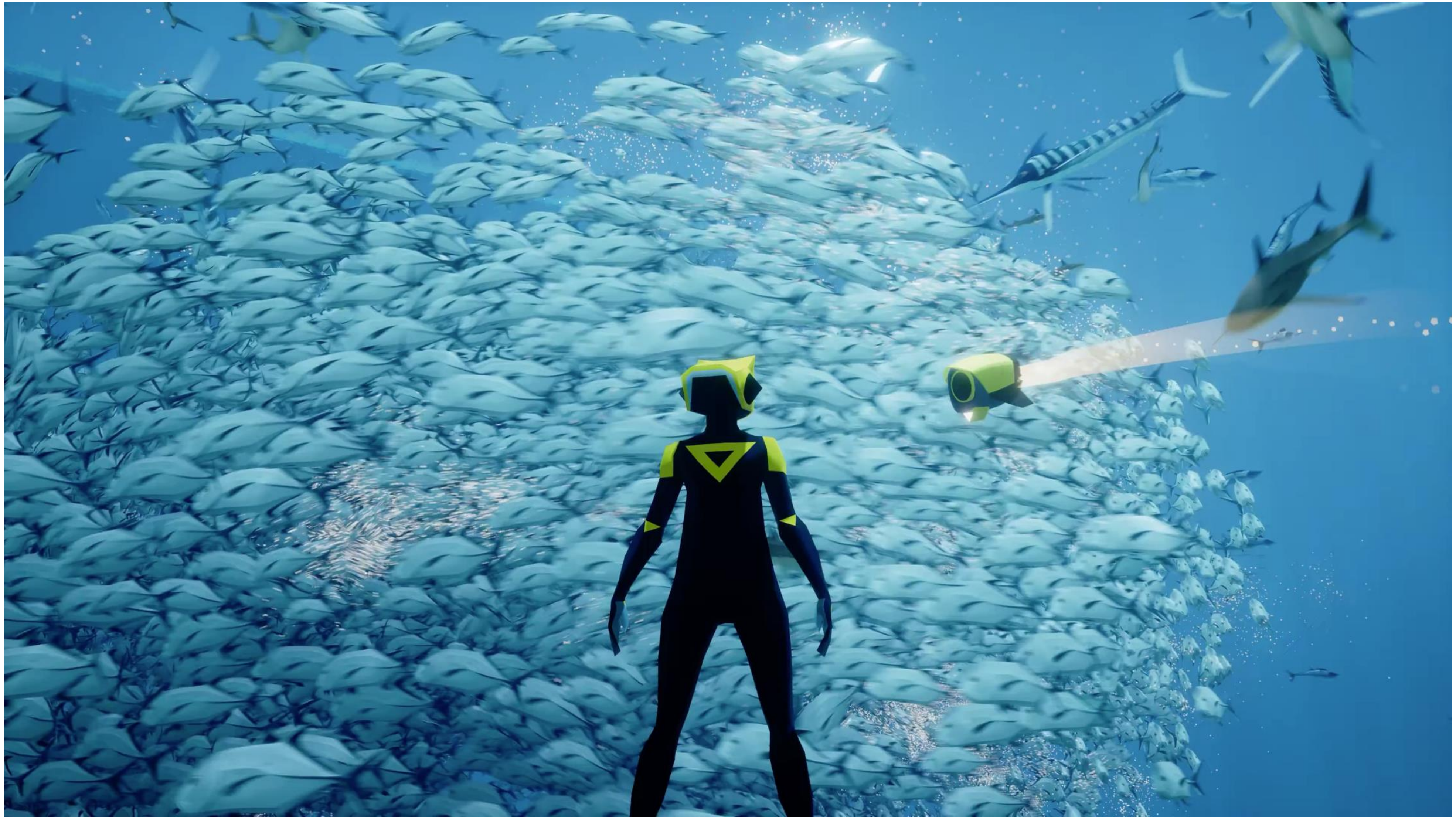


boids

Boids: state of the art

Batman Returns
1992

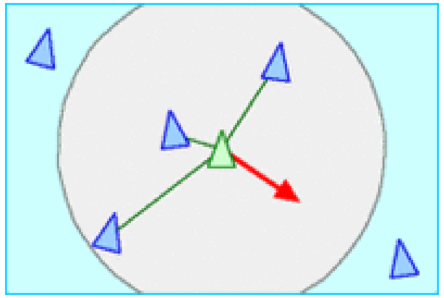
Boids: state of the art



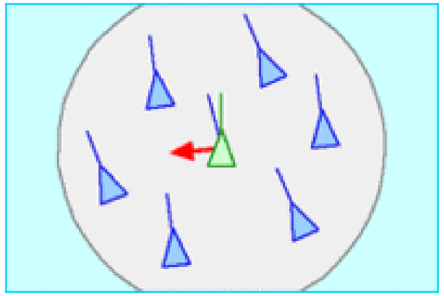
Abzu
2016

Reynold's rules for flocking

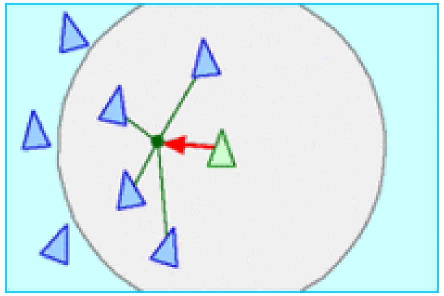
agents must:



1) avoid collisions with each other (separation)



2) match their velocity to nearby mates (alignment)



3) move towards the center of mass of its neighbors (cohesion)

The Physics community

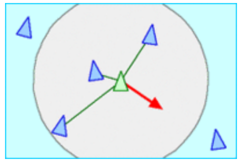
What are the critical transitions?

$$x_i(t+1) = x_i(t) + \Delta t \, v \begin{bmatrix} \cos\theta_i(t) \\ \sin\theta_i(t) \end{bmatrix}$$

position

constant velocity

interaction term



$$\theta_i = \langle \theta_j \rangle_{\|x_j - x_i\| < r}$$

T Vicsek, *Physical Review Letters* (1995).

The Physics community

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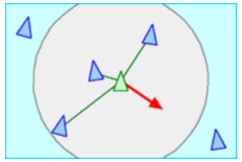

interaction term

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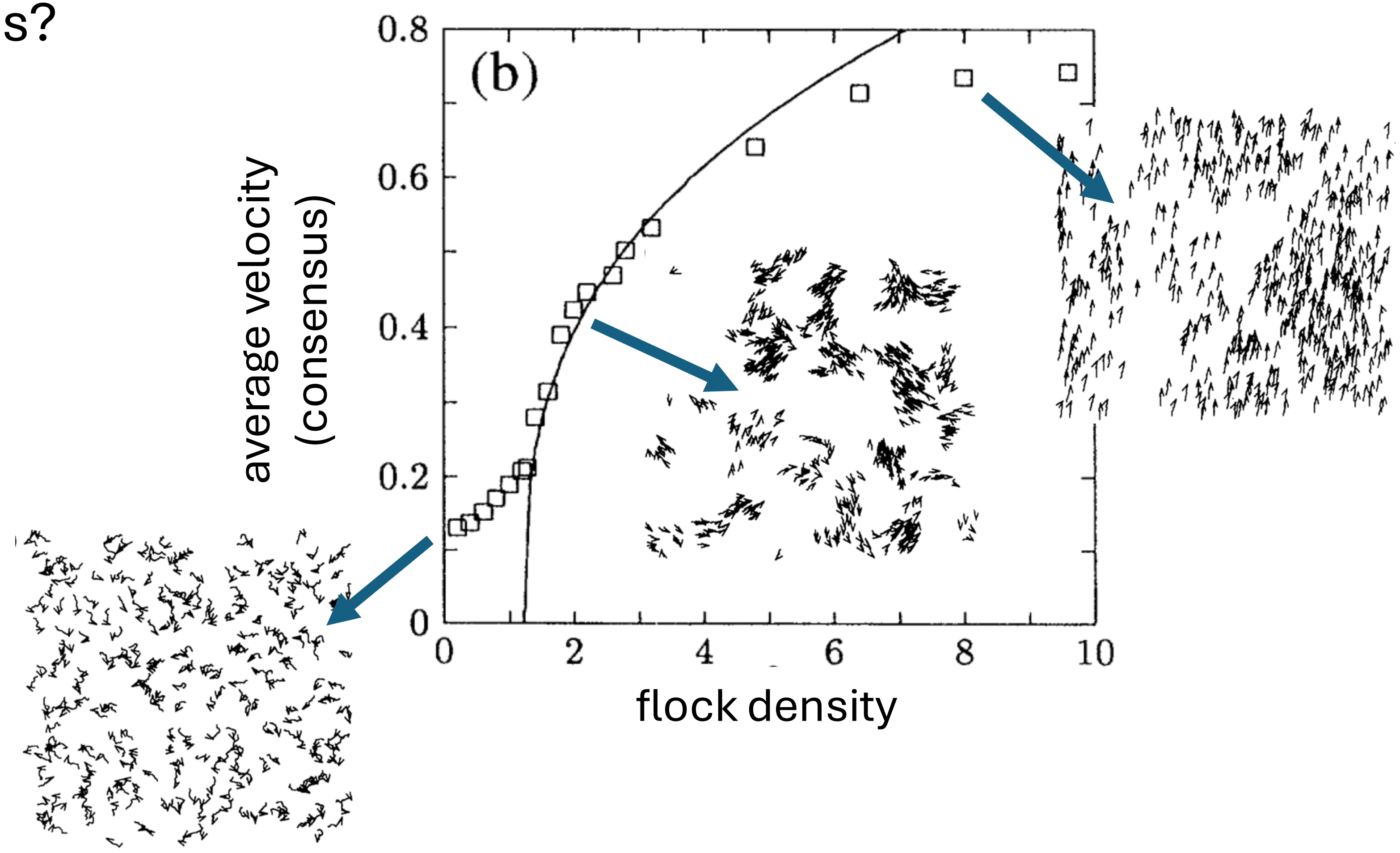

T Vicsek, *Physical Review Letters* (1995).

The Engineering community

Which conditions lead to stability?

$$\dot{q}_i = p_i$$

$$m_i \dot{p}_i = f(q_i, p_i) + \sum_j A_{ij}(t) \left[\left(q_j - q_i\right) + \left(p_j - p_i\right) \right]$$

self dynamics

adjacency matrix of a time-varying graph $\mathcal{G}(t)$

The Engineering community

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self dynamics

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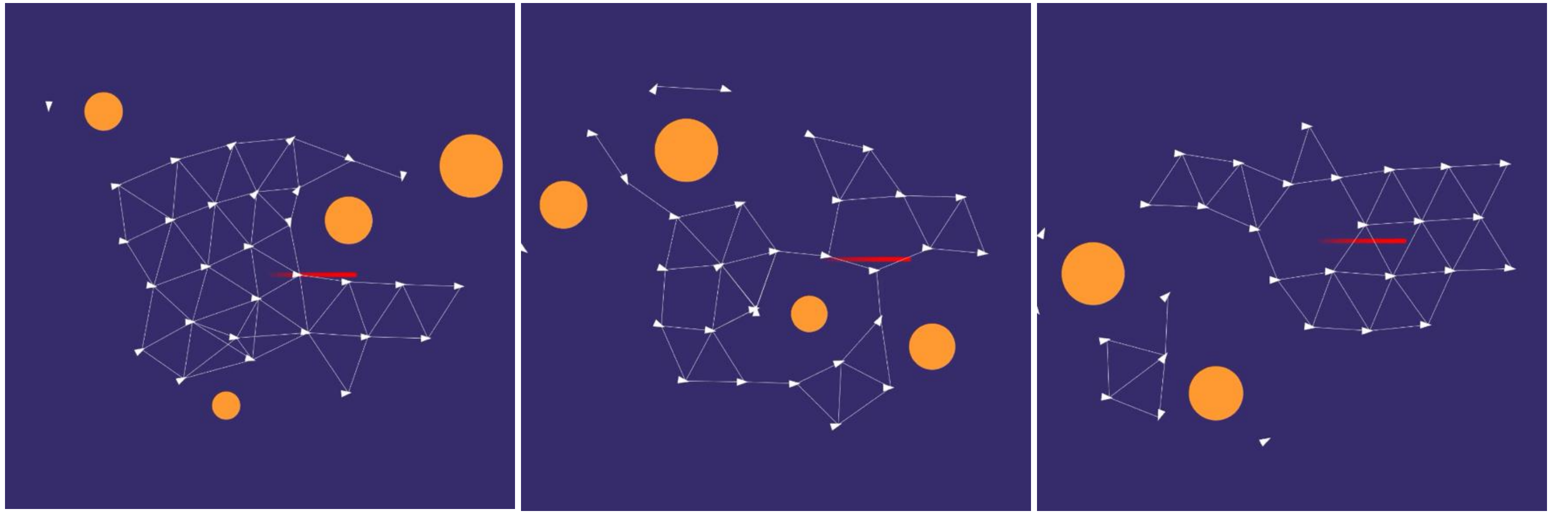
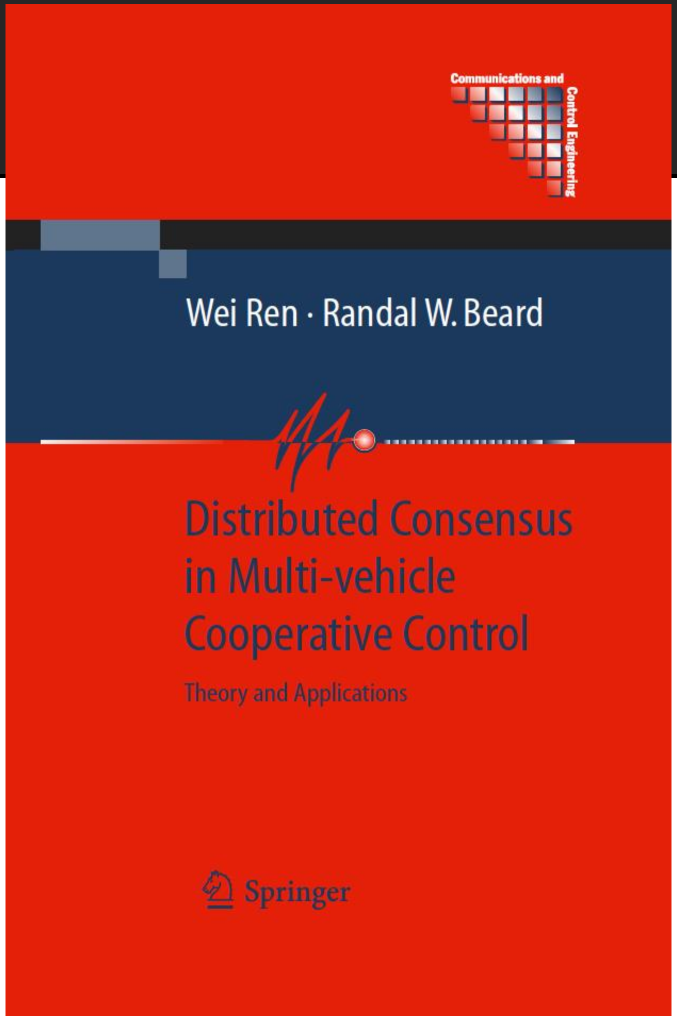
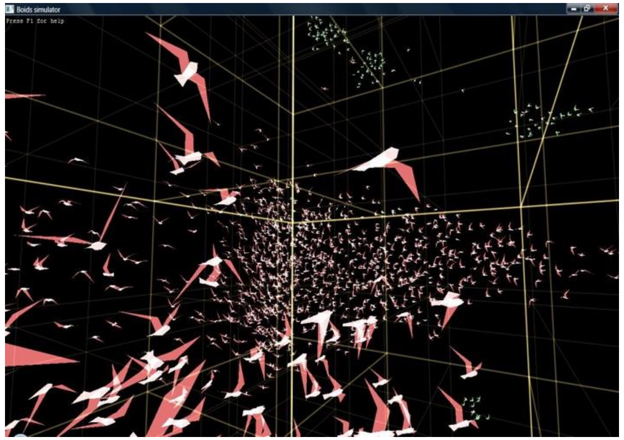




Example:
Consensus is achieved iff $\cup_k \mathcal{G}(t_k)$ has a directed spanning tree.

What are the mechanisms?



boids, phase transition model
(phenomenological)

$$\dot{x}_i = f(x_i) + \text{interaction}$$

drones, autonomous vehicles
(first principles)

$$\dot{q}_i = p_i$$
$$m_i \dot{p}_i = f(q_i, p_i) + \text{interaction}$$

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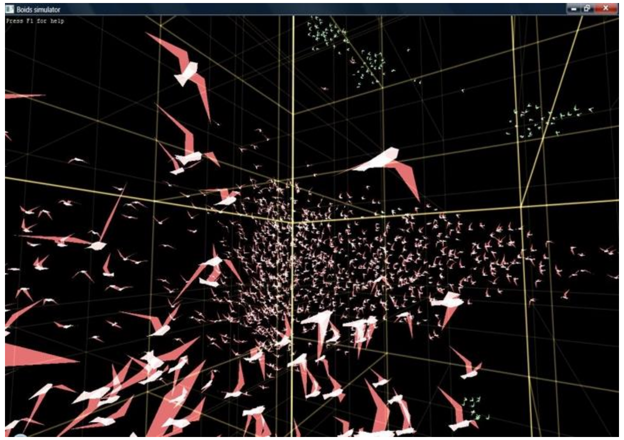

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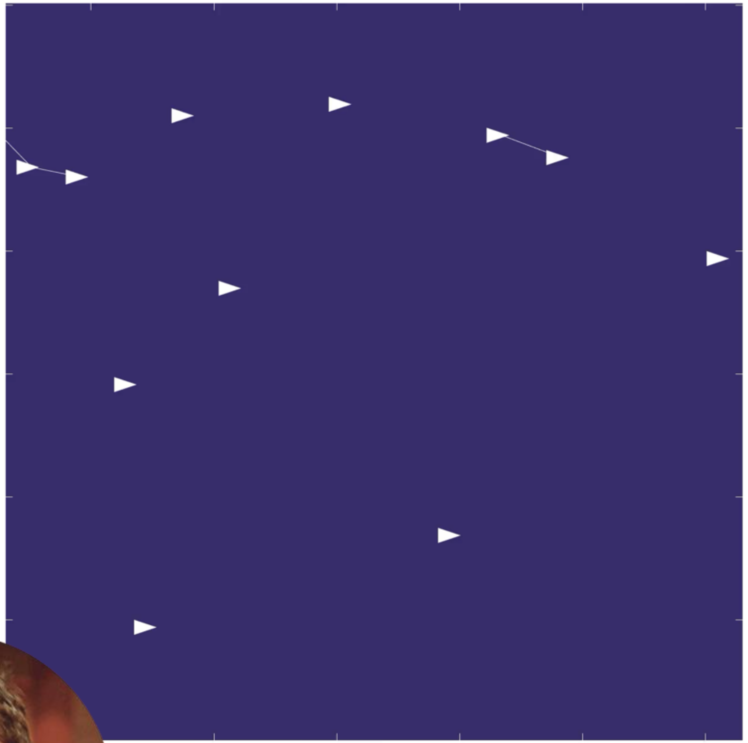
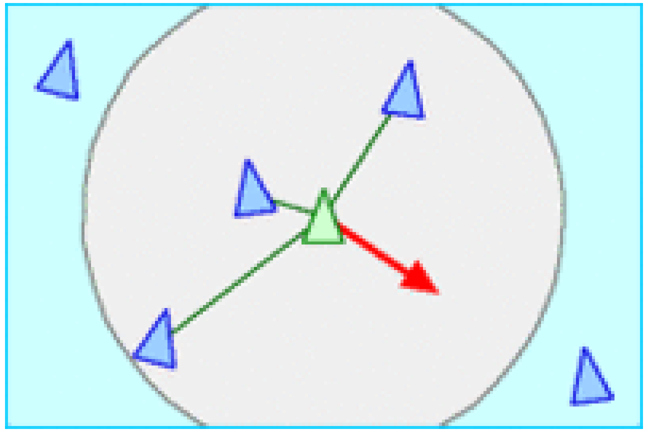
The rules of computation



Iain Couzin, interview at *Quanta Magazine*

If we use advanced imaging tools to quantify, to measure, these waves of turning, it results in a wave of propagation that's around 10 times faster than the maximum speed of the predator itself. So individuals can respond to a predator that they don't even see.

What does biology tell us?



How agents interact with their neighborhood?

Size of the neighborhood depends on...

distance? (metric distance)

number of peers? (topological distance)

PNAS

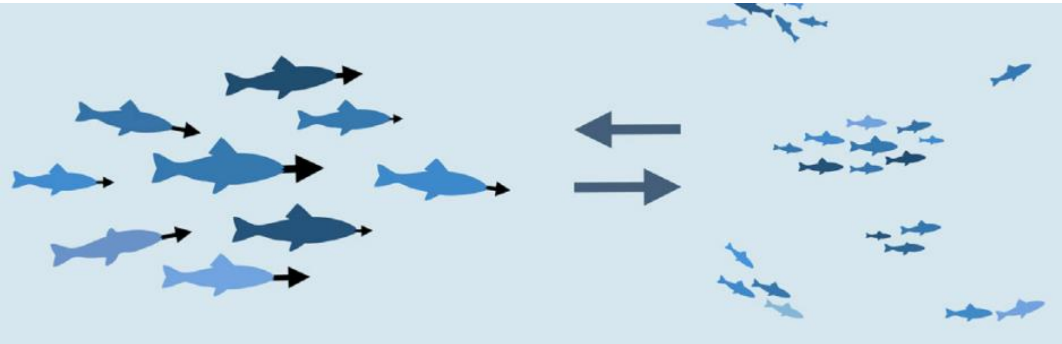
Revealing the hidden networks of interaction in mobile animal groups allows prediction of complex behavioral contagion

Sara Brin Rosenthal[a,1], Colin R. Twomey[b,1], Andrew T. Hartnett[a], Hai Shan Wu[b], and Iain D. Couzin[b,c,d,2]

Departments of [a]Physics and [b]Ecology and Evolutionary Biology, Princeton University, Princeton, NJ 08544; [c]Department of Collective Behaviour, Max Planck Institute for Ornithology, D-78547 Konstanz, Germany; and [d]Chair of Biodiversity and Collective Behavior, Department of Biology, University of Konstanz, D-78547 Konstanz, Germany

Edited by Gene E. Robinson, University of Illinois at Urbana–Champaign, Urbana, IL, and approved February 24, 2015 (received for review October 22, 2014)

What does biology tell us?

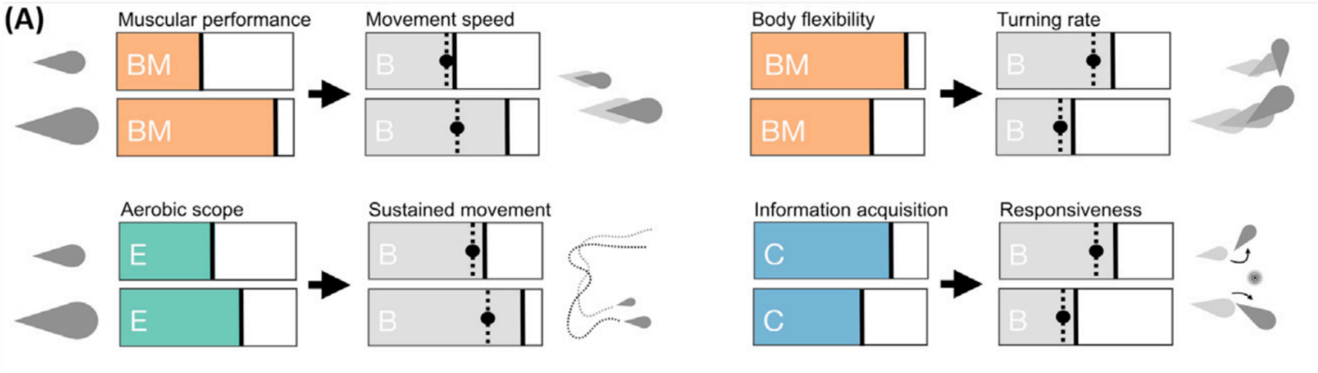


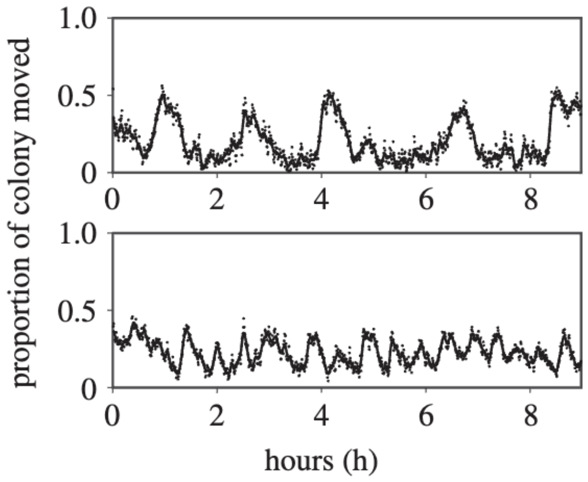
CellPress REVIEWS

Review

The Role of Individual Heterogeneity in Collective Animal Behaviour

Jolle W. Jolles,[1,2,3,7,@,*] Andrew J. King,[4,5] and Shaun S. Killen[6]





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Research

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Cite this article: Doering GN, Drawert B, Lee C, Pruitt JN, Petzold LR, Dalnoki-Veress K. 2022 Noise resistant synchronization and collective rhythm switching in a model of animal group locomotion. *R. Soc. Open Sci.* **9**: 211908. https://doi.org/10.1098/rsos.211908

Noise resistant synchronization and collective rhythm switching in a model of animal group locomotion

Grant Navid Doering[1], Brian Drawert[3], Carmen Lee[2], Jonathan N. Pruitt[1], Linda R. Petzold[4,5] and Kari Dalnoki-Veress[2]

Yet, the Engineering community...

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

A tool for analysis and synthesis of heterogeneous multi-agent systems under rank-deficient coupling☆

Jin Gyu Lee [a], Hyungbo Shim [b,*]

[a] Control Group, Department of Engineering, University of Cambridge, Cambridge, United Kingdom
[b] ASRI, Department of Electrical and Computer Engineering, Seoul National University, Seoul, Republic of Korea

Consensus of heterogeneous multi-agent systems

Y. Zheng[1] Y. Zhu[1] L. Wang[2]

[1]Center for Complex Systems, School of Mechano-electronic Engineering, Xidian University, Xi'an 710071, People's Republic of China
[2]Center for Systems and Control, College of Engineering and Key Laboratory of Machine Perception (Ministry of Education), Peking University, Beijing 100871, People's Republic of China
E-mail: zhengyuanshi2005@163.com

Contents lists available at ScienceDirect

Annual Reviews in Control

journal homepage: www.elsevier.com/locate/arcontrol

Cooperative control of heterogeneous multi-agent systems under spatiotemporal constraints☆

Fei Chen *, Mayank Sewlia, Dimos V. Dimarogonas

Division of Decision and Control Systems, KTH Royal Institute of Technology, SE-100 44, Stockholm, Sweden

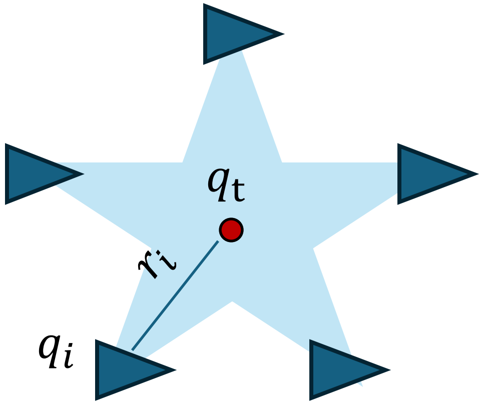
Take-home message

Consensus can be achieved and enhanced not despite, but **because of heterogeneity**.

Flocking model for target tracking

$$\dot{\boldsymbol{q}}_i = \boldsymbol{p}_i,$$

$$m_i \dot{\boldsymbol{p}}_i = \dot{\boldsymbol{p}}_{\mathrm{t}} - b_i\left(\boldsymbol{q}_i - \boldsymbol{q}_{\mathrm{t}} - \boldsymbol{r}_i\right) - \gamma c_i\left(\boldsymbol{p}_i - \boldsymbol{p}_{\mathrm{t}}\right) + \sum_{j=1}^{N} A_{ij}(t)\Big[(\boldsymbol{q}_j - \boldsymbol{r}_j) - (\boldsymbol{q}_i - \boldsymbol{r}_i) + \gamma(\boldsymbol{p}_j - \boldsymbol{p}_i)\Big]$$



pre-specified formation: $\boldsymbol{q}_i(t) \to \boldsymbol{q}_{\mathrm{t}}(t) + \boldsymbol{r}_i$

trajectory tracking: $\boldsymbol{p}_i(t) \to \boldsymbol{p}_{\mathrm{t}}(t)$ as $t \to \infty$

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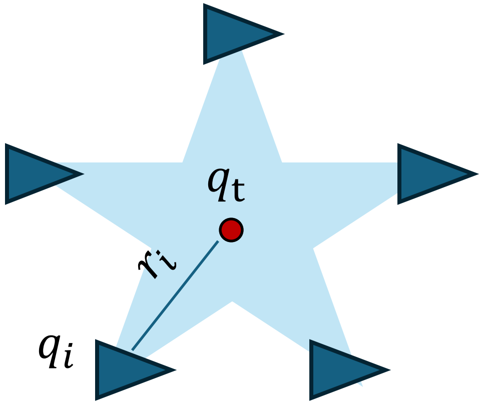

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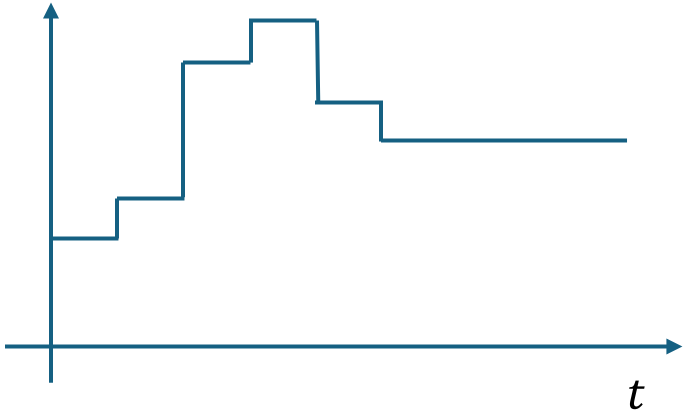
adjacency matrix: $\tilde{A}_{ij}(t) = K/(\rho^2 + \|\boldsymbol{q}_i(t) - \boldsymbol{q}_j(t)\|^2)^\beta$

all-to-all, weighted,
time dependent,
piecewise constant



Flocking model for target tracking

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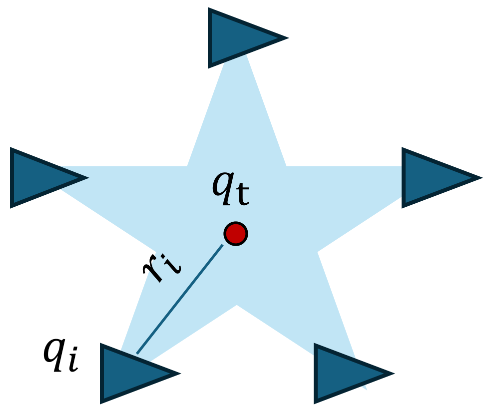
$$+ \sum_{j=1}^{N} A_{ij}(t) \Big[(\boldsymbol{q}_j - \boldsymbol{r}_j) - (\boldsymbol{q}_i - \boldsymbol{r}_i) + \gamma (\boldsymbol{p}_j - \boldsymbol{p}_i) \Big]$$



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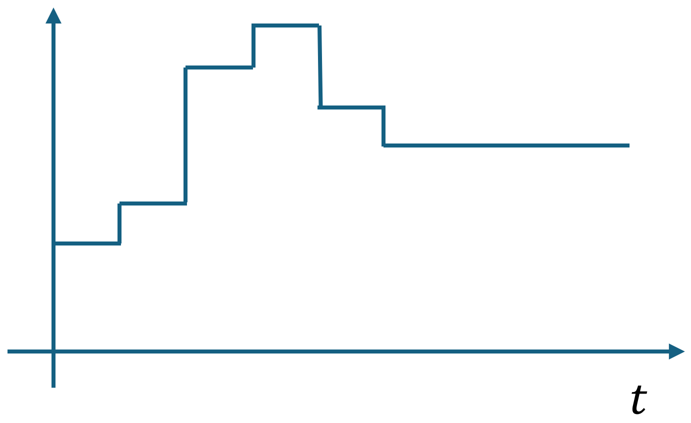
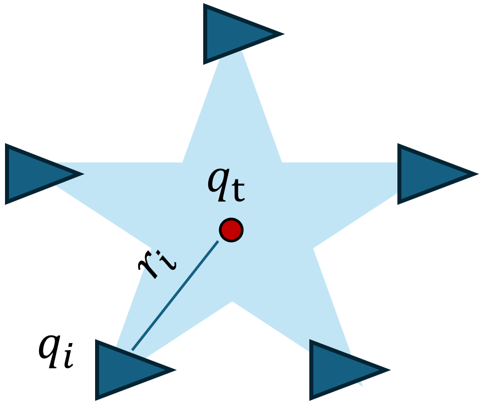


Dynamics of the tracking error

LTV system $[\boldsymbol{e}_{q,i},\ \boldsymbol{e}_{p,i}] = [\boldsymbol{q}_i - (\boldsymbol{q}_\mathrm{t} + \boldsymbol{r}_i),\ \boldsymbol{p}_i - \boldsymbol{p}_\mathrm{t}]$



$$\begin{bmatrix} \dot{\boldsymbol{e}}_q \\ \dot{\boldsymbol{e}}_p \end{bmatrix} = \underbrace{\begin{bmatrix} 0_{Nm} & I_{Nm} \\ -I_m \otimes M(B+L(t)) & -I_m \otimes \gamma M(C+L(t)) \end{bmatrix}}_{J(t)} \underbrace{\begin{bmatrix} \boldsymbol{e}_q \\ \boldsymbol{e}_p \end{bmatrix}}_{\boldsymbol{e}(t)}$$

$$B = \begin{bmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_N \end{bmatrix} \qquad M = \begin{bmatrix} m_1^{-1} & & & \\ & m_2^{-1} & & \\ & & \ddots & \\ & & & m_N^{-1} \end{bmatrix} \qquad C = \begin{bmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_N \end{bmatrix}$$

$L(t)$: Laplacian matrix

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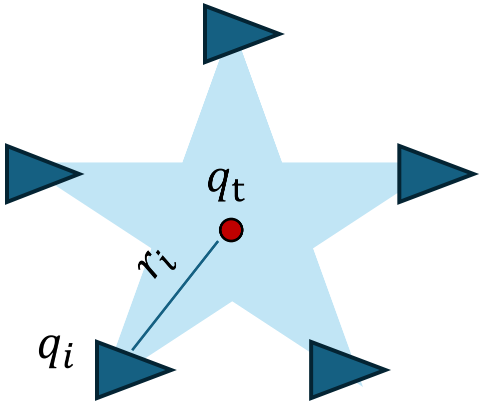
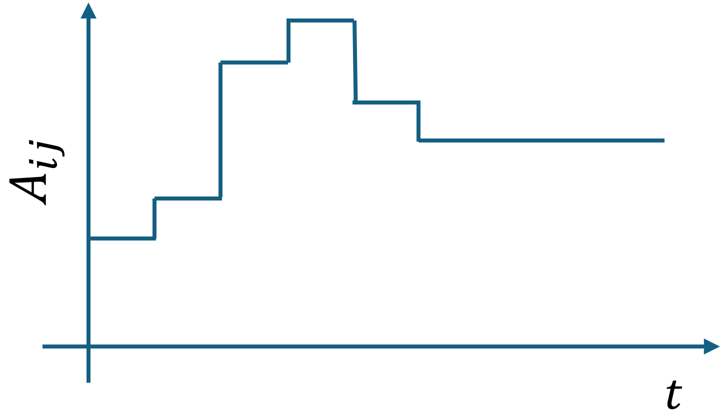


$$\|\boldsymbol{e}(t)\| \le \eta \exp\left\{\sum_{k=0}^{t/T} \Lambda_{\max}(J(t_k))T\right\}\|\boldsymbol{e}(0)\|$$



Optimal flocking dynamics

$$\|\boldsymbol{e}(t)\| \leq \eta \exp\left\{\sum_{k=0}^{t/T} \Lambda_{\max}(J(t_k))T\right\}\|\boldsymbol{e}(0)\|$$

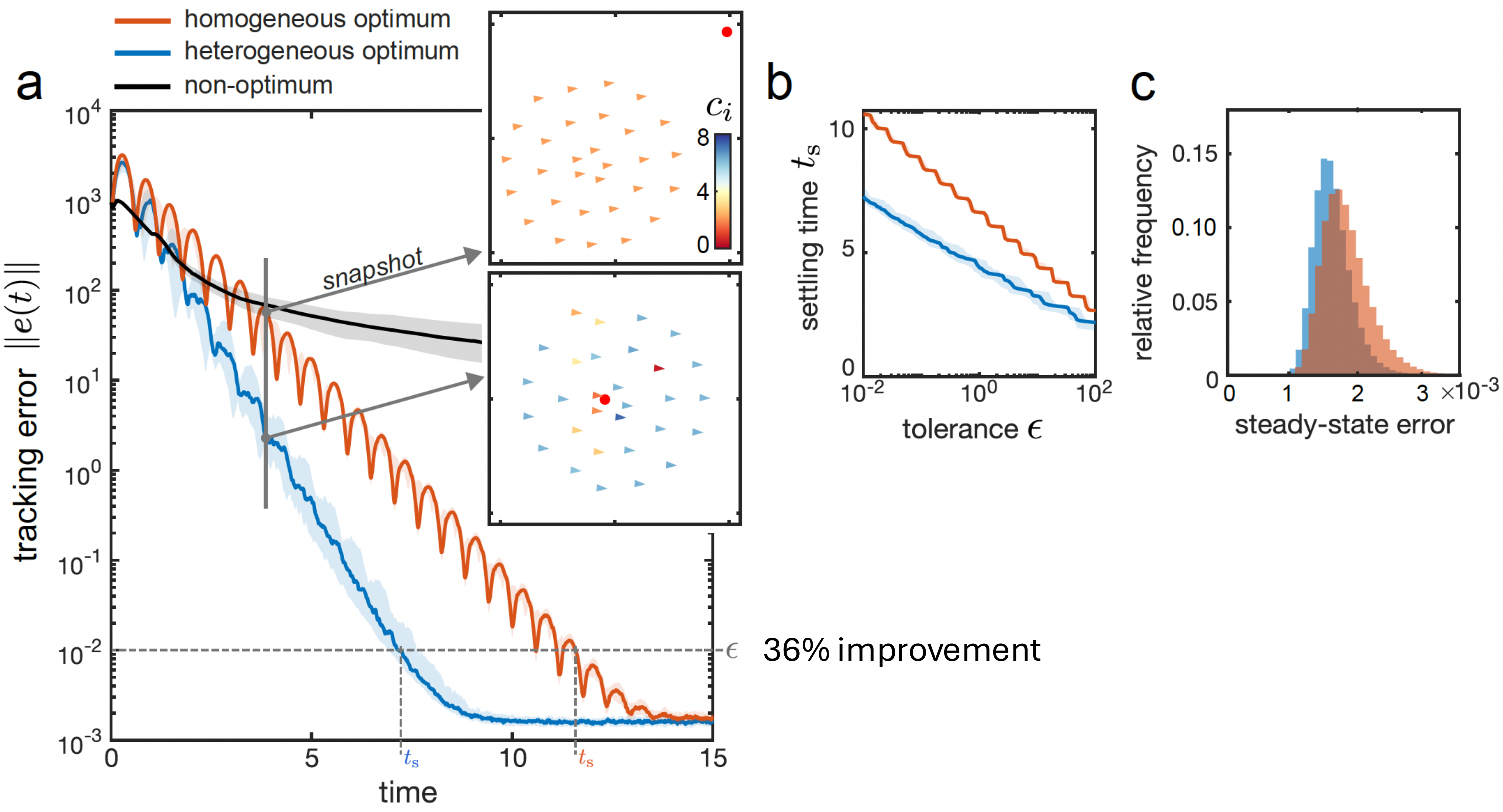
<u>Optimal control procedure</u>

$$\begin{aligned} \min_{\boldsymbol{b},\boldsymbol{c}} \quad & \Lambda_{\max}(J(t_k)), \\ \text{s.t.} \quad & 0 < \boldsymbol{b} \leq b_{\max}, \\ & 0 < \boldsymbol{c} \leq c_{\max}, \end{aligned}$$

solved in "real time" at each time interval $[t_k, t_k + T]$

1. optimal flocks of *homogeneous* agents, where parameters are optimized subject to the constraint that all agents have identical gains, i.e., $\boldsymbol{b}^{(k)} = [b^{(k)}, \ldots, b^{(k)}]$ and $\boldsymbol{c}^{(k)} = [c^{(k)}, \ldots, c^{(k)}]$;
2. optimal flocks of *heterogeneous* agents, where gains are optimized independently for each agent, i.e., $\boldsymbol{b}^{(k)} = [b_1^{(k)}, \ldots, b_N^{(k)}]$ and $\boldsymbol{c}^{(k)} = [c_1^{(k)}, \ldots, c_N^{(k)}]$.

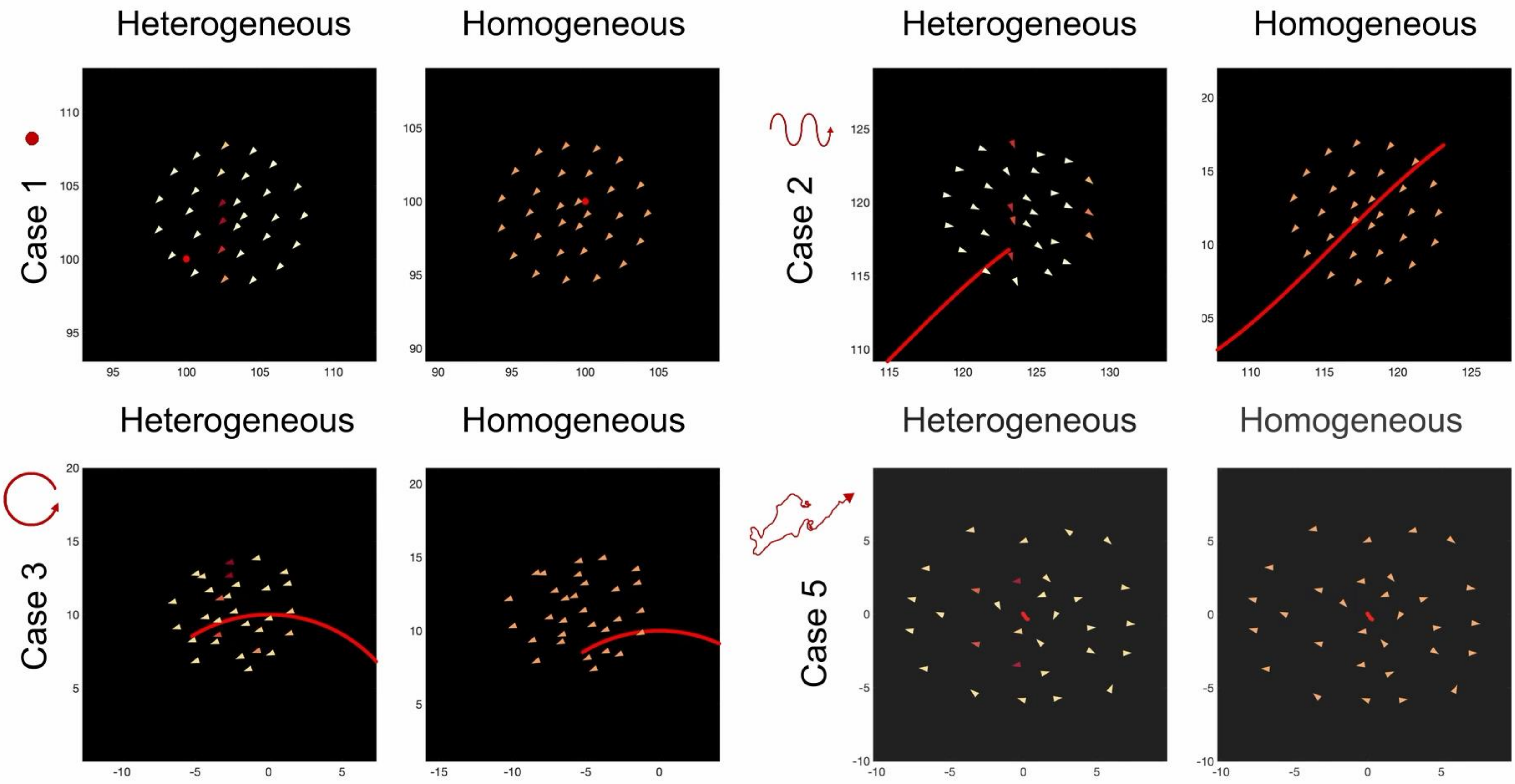
Heterogeneous vs homogeneous flocking



36% improvement

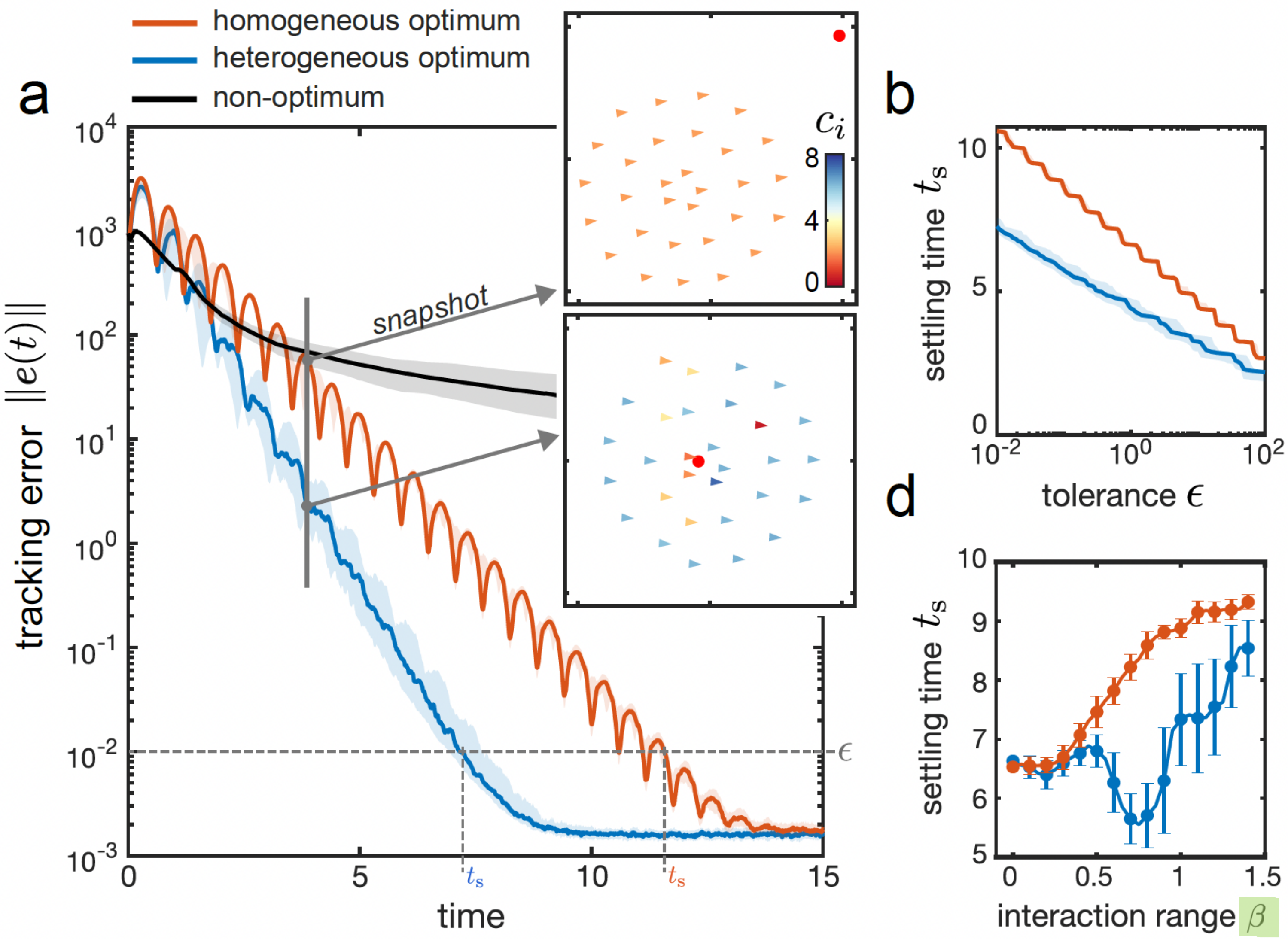
Supplementary Movie 1

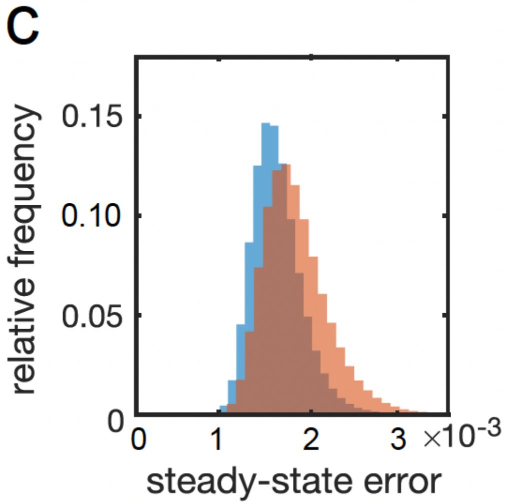
Target Tracking and Flock Formation



Heterogeneous
Homogeneous
Heterogeneous
Homogeneous
Case 1
Case 2
Heterogeneous
Homogeneous
Heterogeneous
Homogeneous
Case 3
Case 5

Heterogeneous vs homogeneous flocking

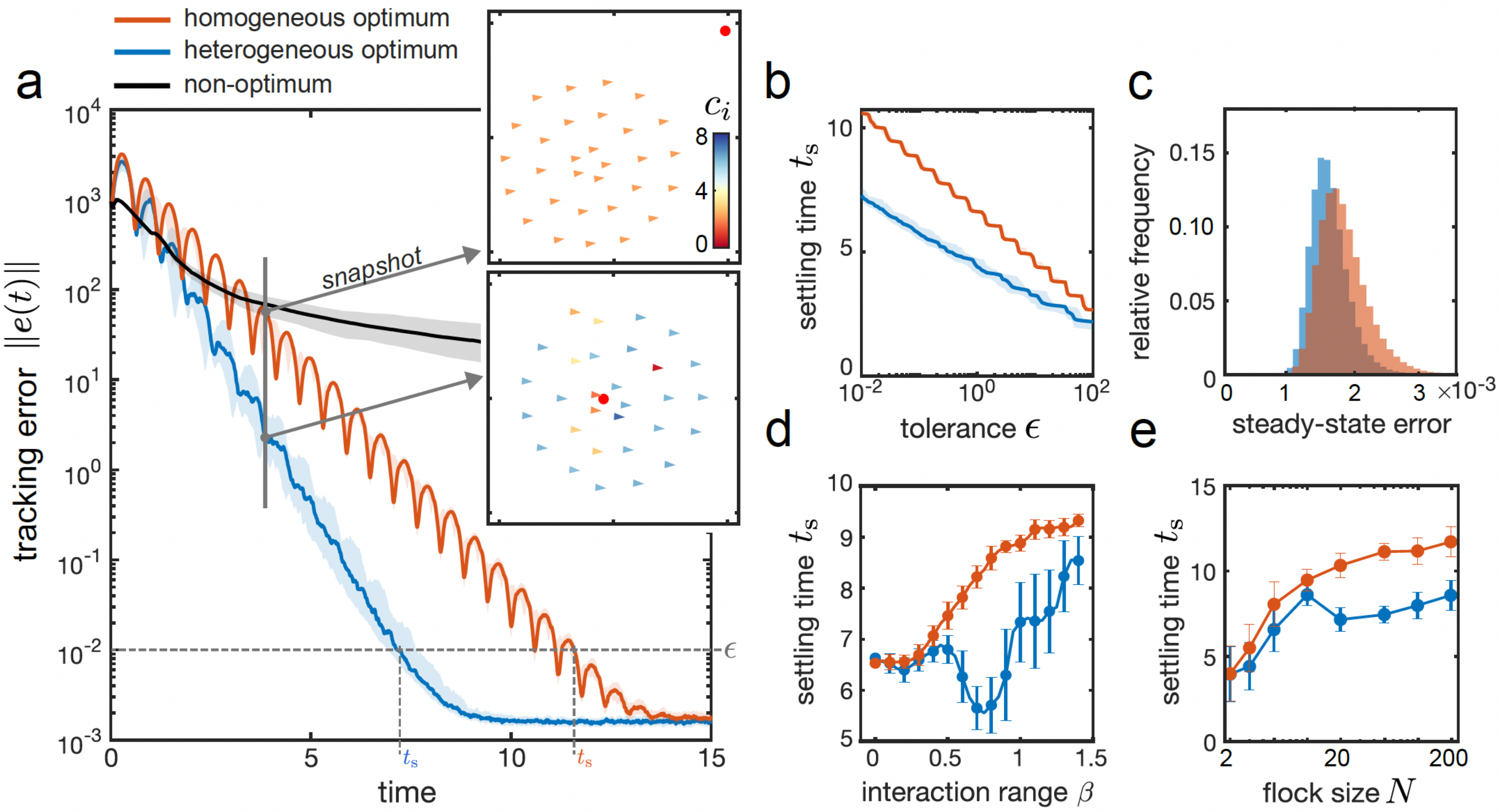




$$\tilde{A}_{ij}(t) = K/(\rho^2 + \|\boldsymbol{q}_i(t) - \boldsymbol{q}_j(t)\|^2)^{\beta}$$

$$B(t) + L(t) = \begin{bmatrix} b_1 + \sum_j A_{1j} & -A_{12} & \cdots & -A_{1N} \\ -A_{21} & b_2 + \sum_j A_{2j} & \cdots & -A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -A_{N1} & -A_{N2} & \cdots & b_N + \sum_j A_{Nj} \end{bmatrix}$$

Heterogeneous vs homogeneous flocking

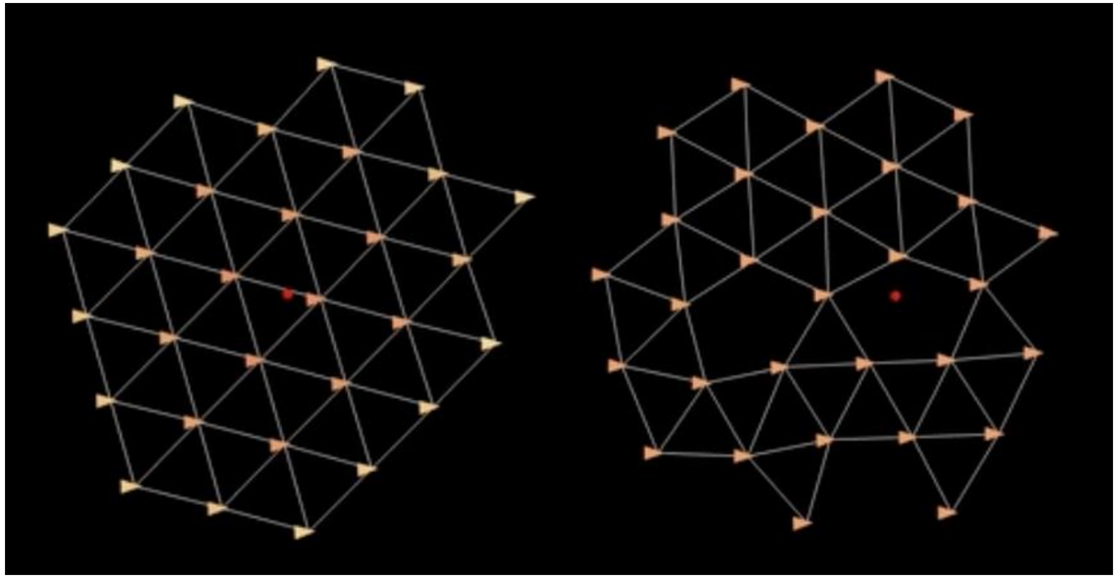


Extension to free-flocking model

$$\dot{\boldsymbol{q}}_i = \boldsymbol{p}_i,$$

$$\dot{\boldsymbol{p}}_i = \boldsymbol{u}_i^\alpha + \boldsymbol{u}_i^\gamma + \boldsymbol{u}_i^\beta,$$

R Olfati-Saber, *IEEE Trans. Automatic Control* (2006).



model complexity

previous | **current**

Extension to free-flocking model

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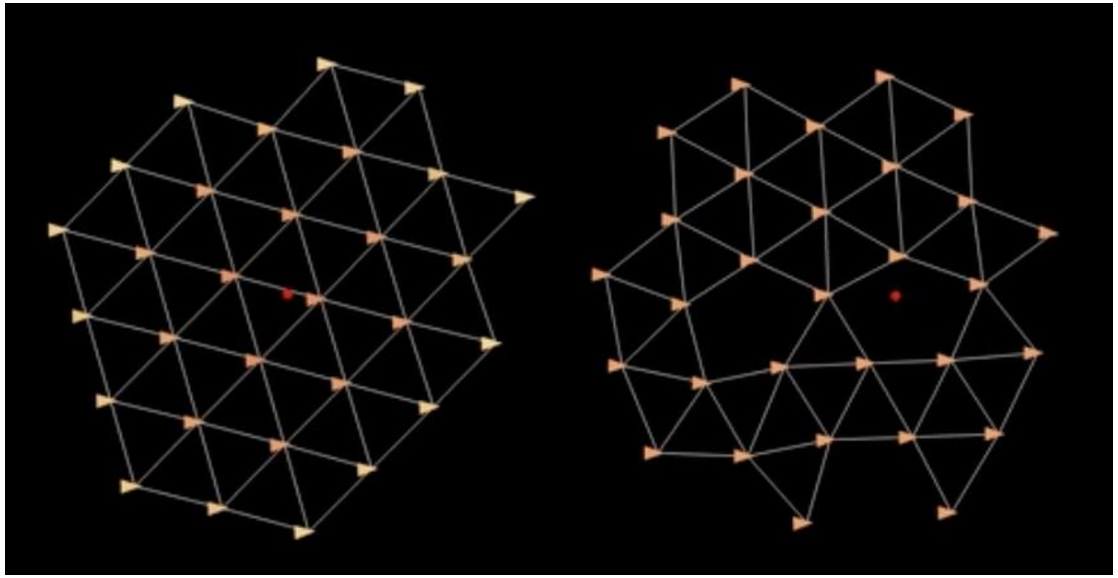
agent-agent interaction:

$$\boldsymbol{u}_i^\alpha = -k_1^\alpha \underbrace{\nabla_{q_i} V(q)}_{} + k_2^\alpha \sum_{j \in \mathcal{N}_i(\boldsymbol{q})} A_{ij}(\boldsymbol{q})(\boldsymbol{p}_j - \boldsymbol{p}_i)$$

$V(\boldsymbol{q}^*) = 0$
iff $\|\boldsymbol{q}_i^* - \boldsymbol{q}_j^*\| = d$



model complexity

previous	current
pre-assigned formation	emergent formation

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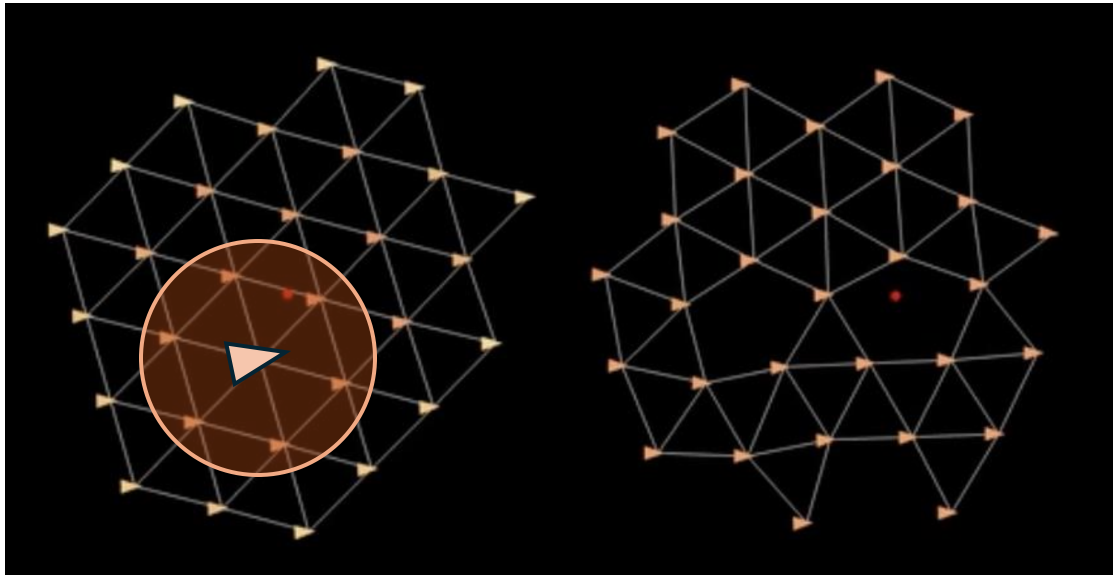


model complexity

previous	**current**
pre-assigned formation	emergent formation
all-to-all, weighted network	sparse, weighted network
piecewise constant adj. matrix	continuous adjacency matrix
linear time-varying dynamics	nonlinear dynamics

Extension to free-flocking model

$$\dot{\boldsymbol{q}}_i = \boldsymbol{p}_i\,,$$
$$\dot{\boldsymbol{p}}_i = \boldsymbol{u}_i^{\alpha} + \boldsymbol{u}_i^{\gamma} + \boldsymbol{u}_i^{\beta}\,,$$

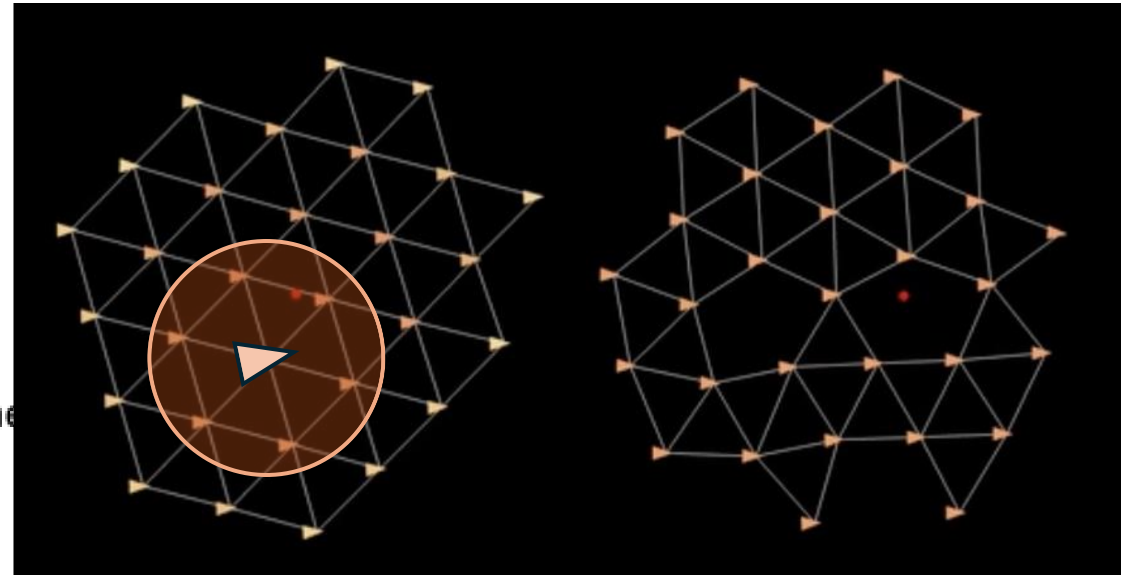
R Olfati-Saber, *IEEE Trans. Automatic Control* (2006).

agent-agent interaction:

$$\boldsymbol{u}_i^{\alpha} = -k_1^{\alpha}\boldsymbol{\nabla}_{\boldsymbol{q}_i}V(\boldsymbol{q}) + k_2^{\alpha}\sum_{j\in\mathcal{N}_i(\boldsymbol{q})} A_{ij}(\boldsymbol{q})(\boldsymbol{p}_j - \boldsymbol{p}_i)$$

agent-target interaction:

$$\boldsymbol{u}_i^{\gamma} = -b_i(\boldsymbol{q}_i - \boldsymbol{q}_{\mathrm{t}}) - c_i(\boldsymbol{p}_i - \boldsymbol{p}_{\mathrm{t}})$$



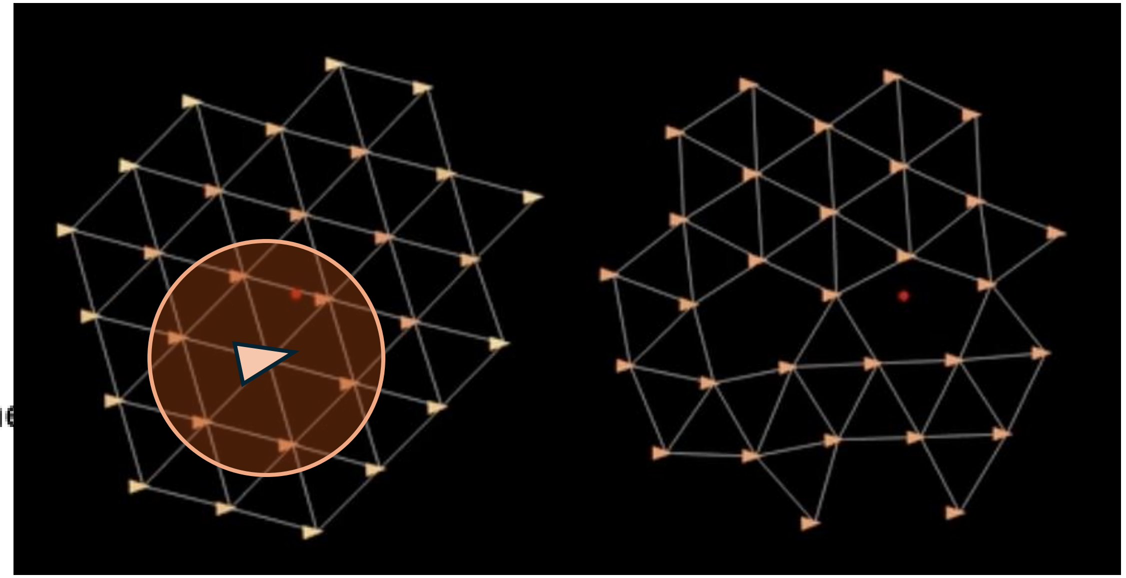
model complexity

previous	**current**
pre-assigned formation	emergent formation
all-to-all, weighted network	sparse, weighted network
piecewise constant adj. matrix	continuous adjacency matrix
time-varying dynamics	nonlinear dynamics

Extension to free-flocking model

$$\dot{\boldsymbol{q}}_i = \boldsymbol{p}_i \,,$$
$$\dot{\boldsymbol{p}}_i = \boldsymbol{u}_i^\alpha + \boldsymbol{u}_i^\gamma + \boldsymbol{u}_i^\beta \,,$$

R Olfati-Saber, *IEEE Trans. Automatic Control* (2006).



agent-agent interaction:

$$\boldsymbol{u}_i^\alpha = -k_1^\alpha \boldsymbol{\nabla}_{\boldsymbol{q}_i} V(\boldsymbol{q}) + k_2^\alpha \sum_{j \in \mathcal{N}_i(\boldsymbol{q})} A_{ij}(\boldsymbol{q})(\boldsymbol{p}_j - \boldsymbol{p}_i)$$

agent-target interaction:

$$\boldsymbol{u}_i^\gamma = -b_i(\boldsymbol{q}_i - \boldsymbol{q}_{\mathrm{t}}) - c_i(\boldsymbol{p}_i - \boldsymbol{p}_{\mathrm{t}})$$

agent-obstacle interaction:

$\boldsymbol{u}_i^\beta = 0$ (... for now)

model complexity

previous	**current**
pre-assigned formation	emergent formation
all-to-all, weighted network	sparse, weighted network
piecewise constant adj. matrix	continuous adjacency matrix
time-varying dynamics	nonlinear dynamics
	3 Reynold's rules

Optimal free flocking

Nonlinear system stability analysis

Lyapunov function: $H(\boldsymbol{e}) = V(\boldsymbol{e}_q) + \frac{1}{2}\bar{\boldsymbol{e}}^\top \bar{\boldsymbol{e}}$

1) $\alpha_1 \|\bar{\boldsymbol{e}}\|^2 \leq H(\boldsymbol{e}) \leq \alpha_2 \|\bar{\boldsymbol{e}}\|^2$

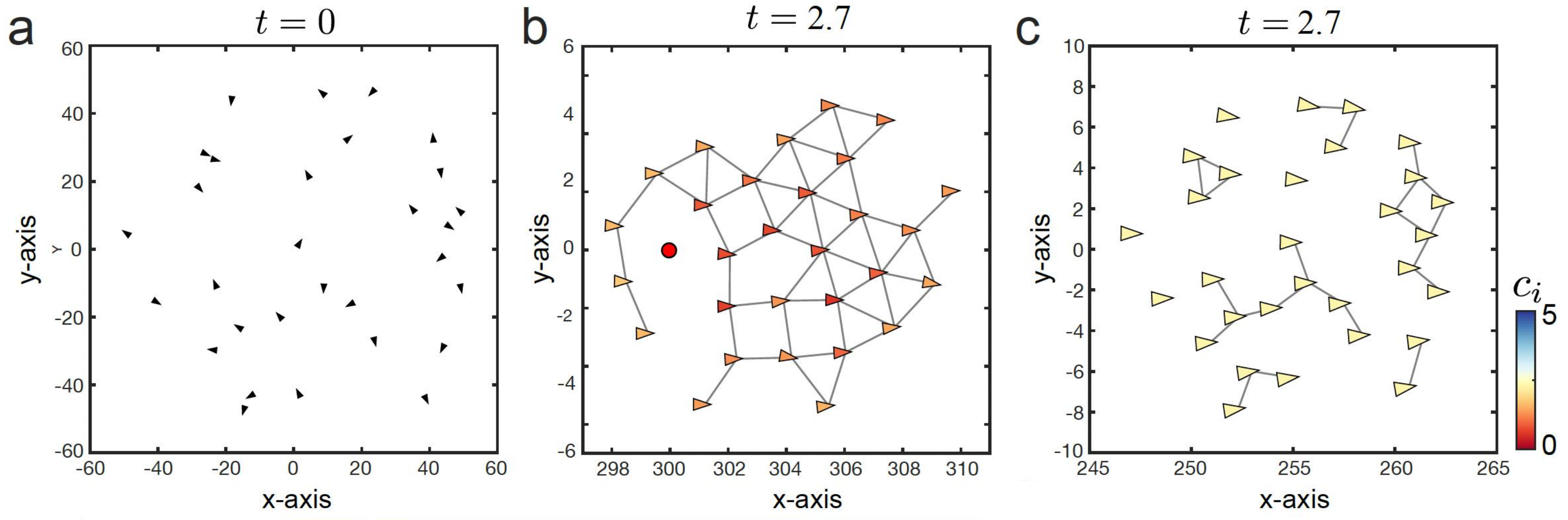
2) $\dot{H}(\boldsymbol{e}, t) = \bar{\boldsymbol{e}}^\top J(t)\bar{\boldsymbol{e}} \leq \Lambda_{\max}(J(t_k))\|\bar{\boldsymbol{e}}\|^2,$ $\quad J(t_k) = \begin{bmatrix} 0_{Nm} & I_{Nm} \\ -I_m \otimes B & -I_m \otimes [C + L(\boldsymbol{q}(t_k))] \end{bmatrix}$

$$\|\boldsymbol{e}(t) - \boldsymbol{e}^*\| \leq \eta \exp\left\{ \frac{\eta_k}{2\alpha_2} \Lambda_{\max}(J(t_k)) T \right\} \|\boldsymbol{e}(t_k) - \boldsymbol{e}^*\|$$

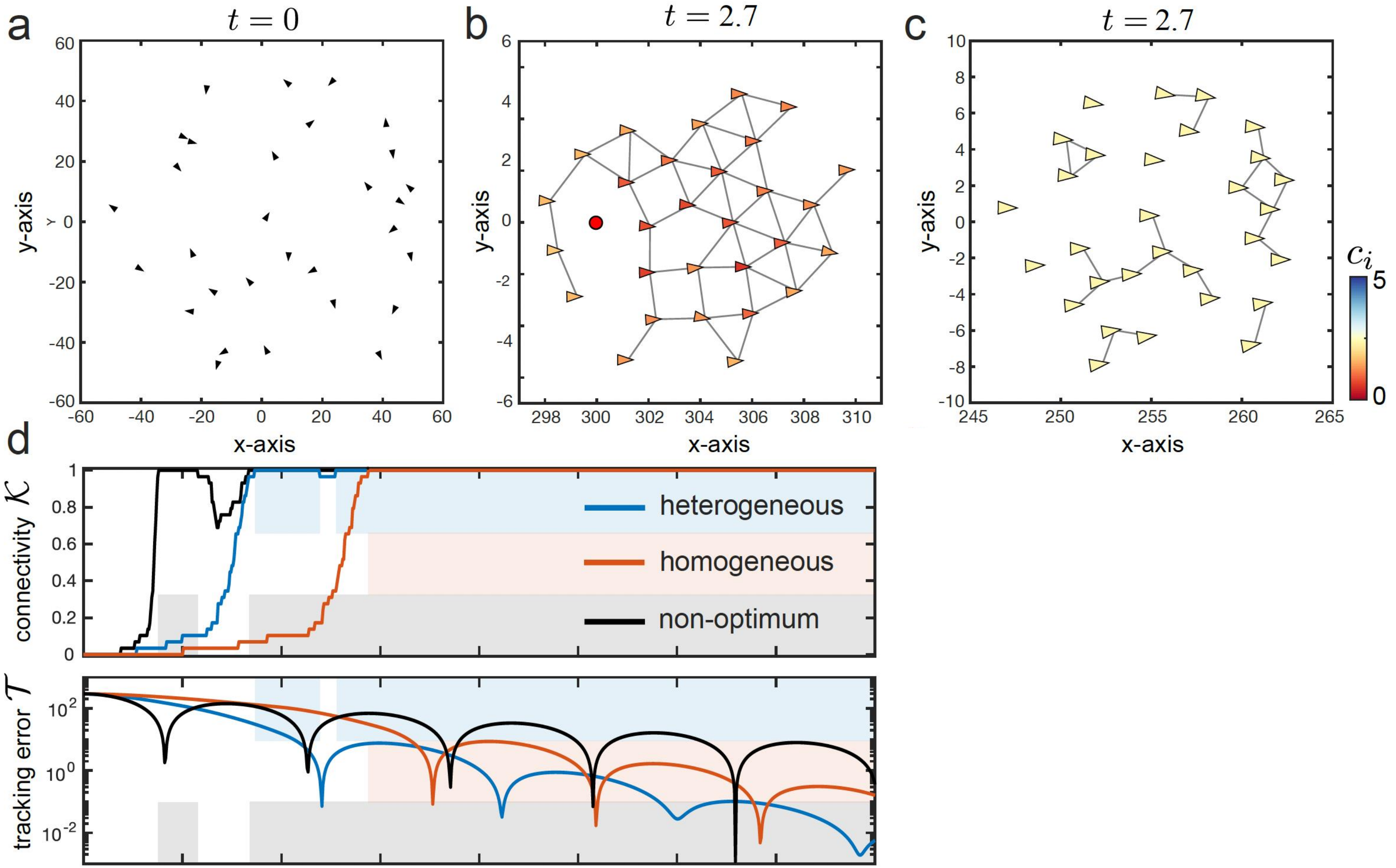
Supplementary Movie 2

Optimal Free Flocking

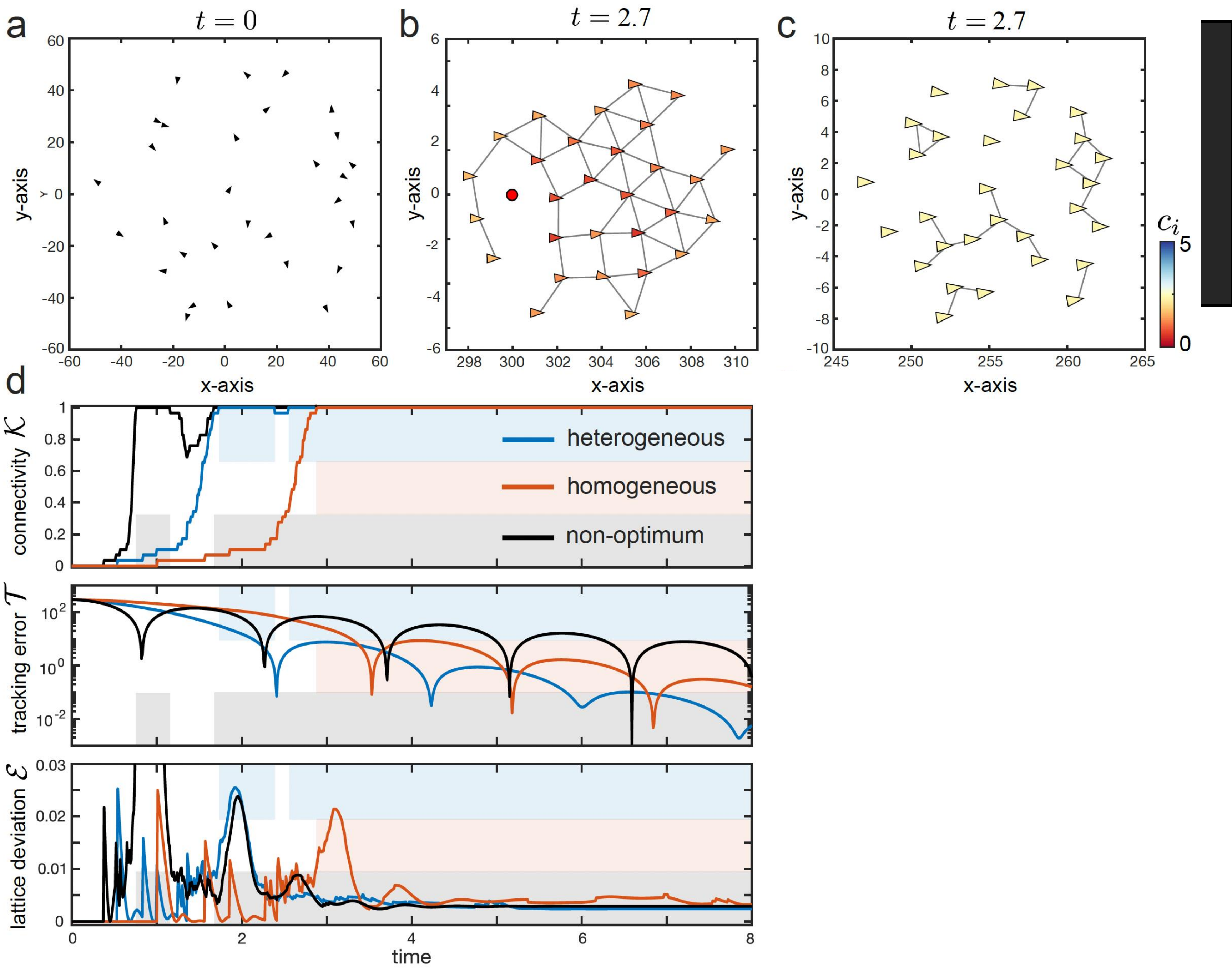
Optimal free flocking

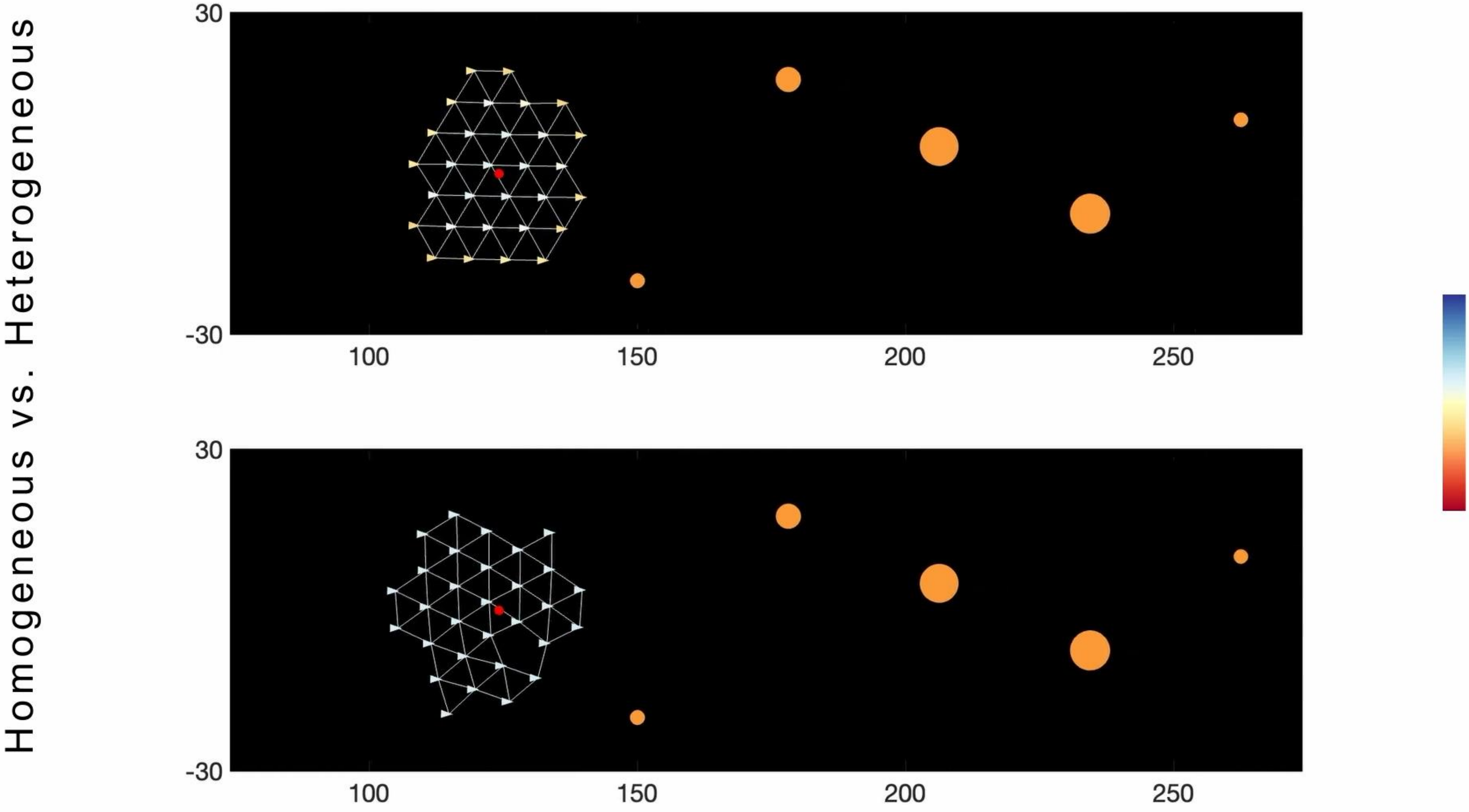


Optimal free flocking

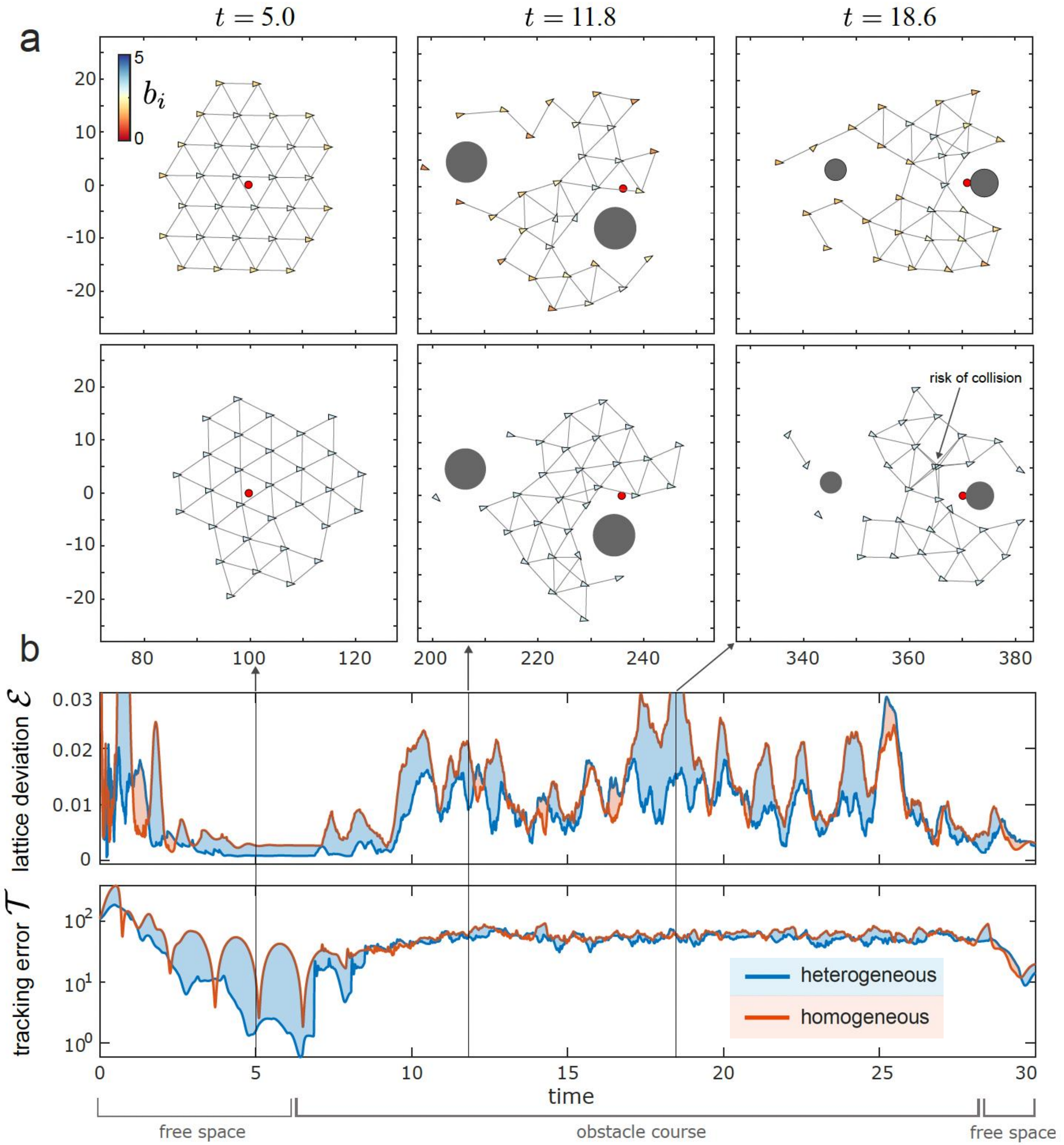


Optimal free flocking




Homogeneous vs. Heterogeneous
30
-30
100
150
200
250
30
-30
100
150
200
250

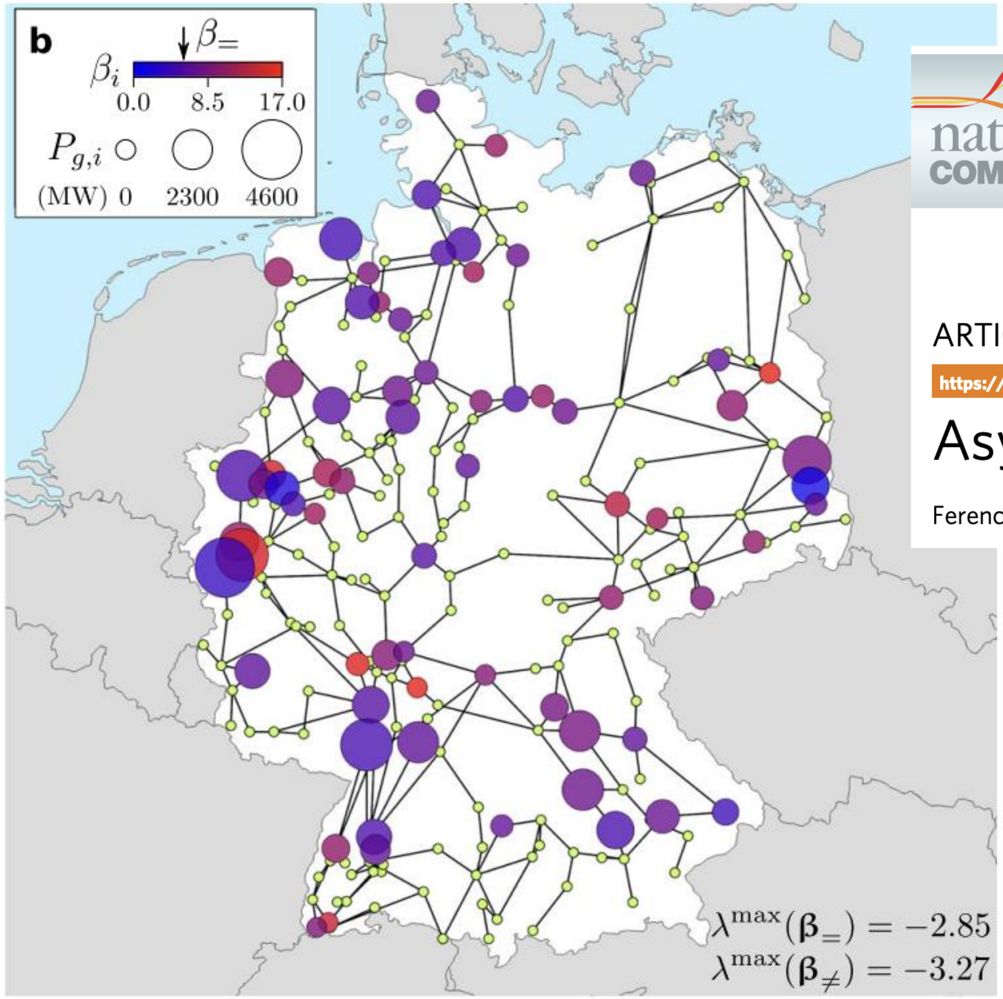
Optimal obstacle maneuvering



Take-home question

Why does **heterogeneity** improve **collective** behavior?

Other research on heterogeneity



nature COMMUNICATIONS

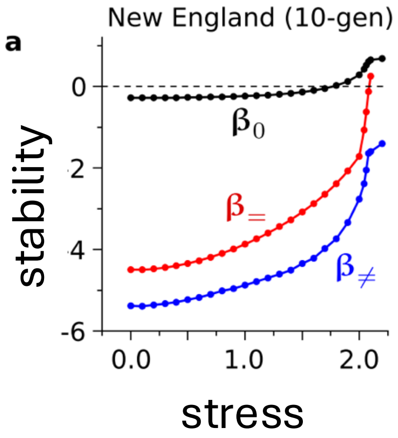
ARTICLE

Check for updates

https://doi.org/10.1038/s41467-021-21290-5 OPEN

Asymmetry underlies stability in power grids

Ferenc Molnar[1,3], Takashi Nishikawa[1,2]✉ & Adilson E. Motter[1,2]

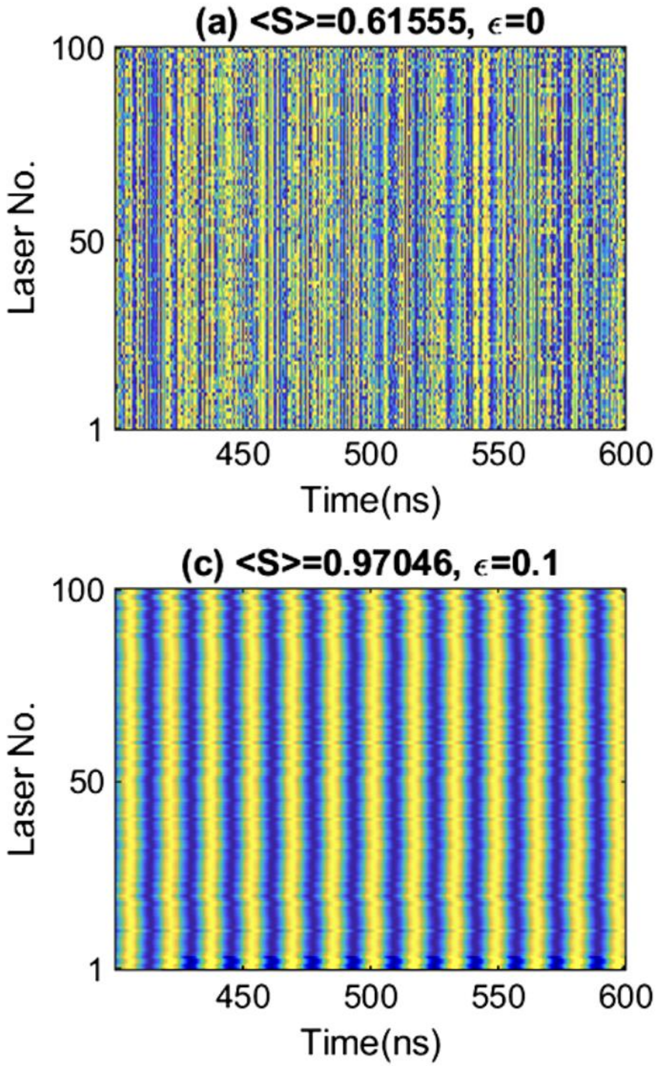


Other research on heterogeneity

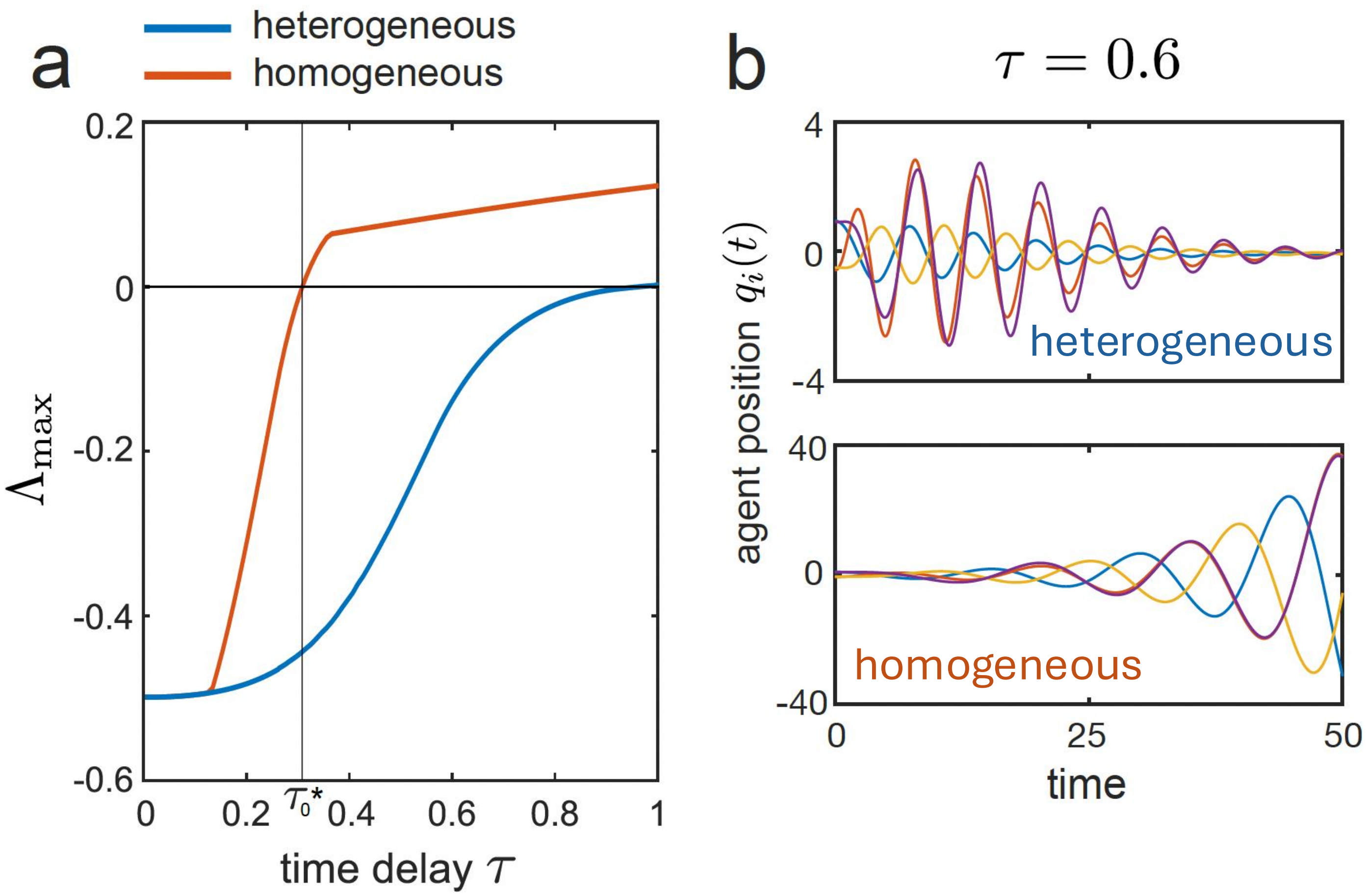
PHYSICAL REVIEW LETTERS **127,** 173901 (2021)

Using Disorder to Overcome Disorder: A Mechanism for Frequency and Phase Synchronization of Diode Laser Arrays

N. Nair,[1,*] K. Hu,[1,†] M. Berrill,[2,‡] K. Wiesenfeld,[3,§] and Y. Braiman[1,4,‖]
[1]*The College of Optics and Photonics (CREOL), University of Central Florida, Orlando, Florida 32816, USA*
[2]*Computer Science and Mathematics Division, Oak Ridge National Laboratory, Oak Ridge, Tennessee 37831, USA*
[3]*School of Physics, Georgia Institute of Technology, Atlanta, Georgia 30332, USA*
[4]*Department of Electrical and Computer Engineering, University of Central Florida, Orlando, Florida 32816, USA*



Other research on heterogeneity

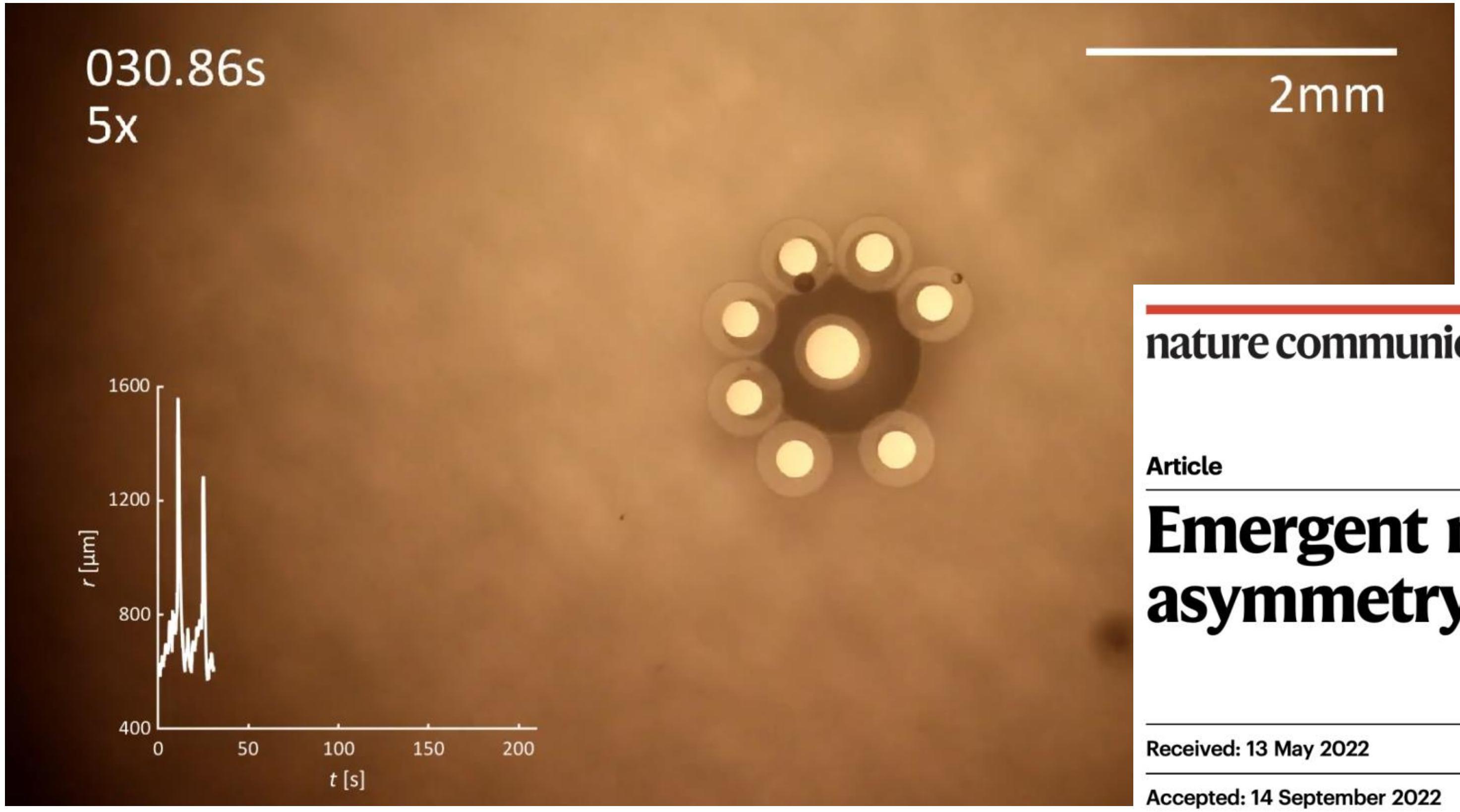


time-delay multi-agent model

$$\dot{\boldsymbol{q}}_i(t) = \boldsymbol{p}_i(t),$$

$$\dot{\boldsymbol{p}}_i(t) = -k_i \left(\sum_{j=1}^{N} L_{ij} \boldsymbol{q}_j(t-\tau) + \sum_{j=1}^{N} L_{ij} \boldsymbol{p}_i(t-\tau) \right)$$

Other research on heterogeneity



nature communications

Article

https://doi.org/10.1038/s41467-022-33396-5

Emergent microrobotic oscillators via asymmetry-induced order

Received: 13 May 2022

Accepted: 14 September 2022

Published online: 13 October 2022

Jing Fan Yang[1,6], Thomas A. Berrueta[2,6], Allan M. Brooks[1], Albert Tianxiang Liu[1,3], Ge Zhang[1], David Gonzalez-Medrano[4], Sungyun Yang[1], Volodymyr B. Koman[1], Pavel Chvykov[5], Lexy N. LeMar[1], Marc Z. Miskin[4], Todd D. Murphey[2] & Michael S. Strano[1]

CENTER FOR ROBOTICS AND BIOSYSTEMS
AT MCCORMICK SCHOOL OF ENGINEERING

Why? An eigenvalue problem

Why? An eigenvalue problem

$$\min_{\boldsymbol{b},\boldsymbol{c}} \quad \Lambda_{\max}(J(t_k)),$$
$$\text{s.t.} \quad 0 < \boldsymbol{b} \leq b_{\max},$$
$$\sout{0 < \boldsymbol{c} \leq c_{\max},}$$

$$J'(\boldsymbol{b}) = \begin{bmatrix} 0_N & I_N \\ -(B + L) & -\gamma(B + L) \end{bmatrix}$$

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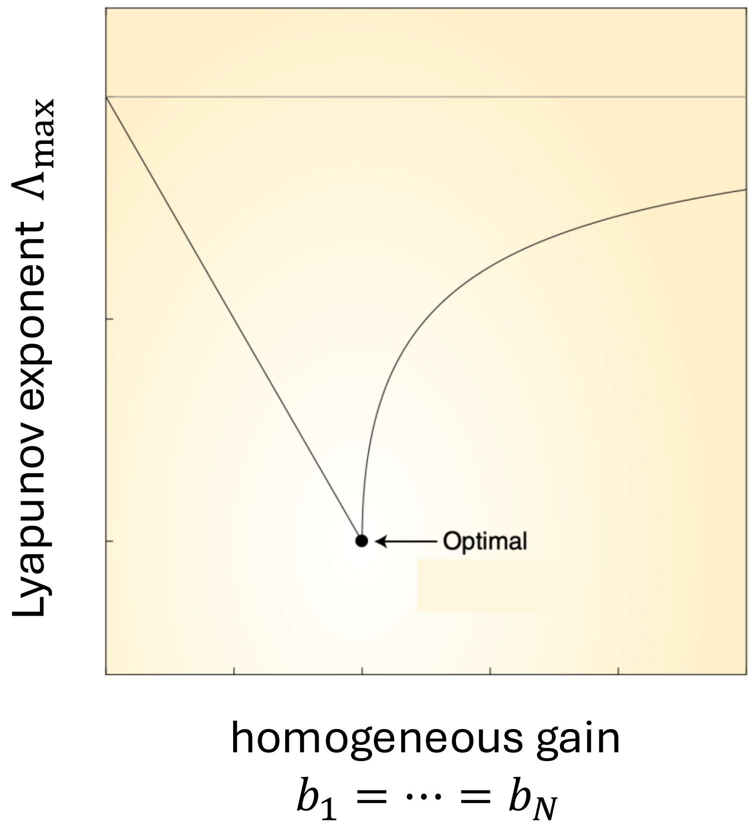
$$b^*_{\text{hom}} = \frac{2}{\gamma^2} - \ell_N + \sqrt{(\ell_N - \ell_1)^2 + \frac{4}{\gamma^4}}$$



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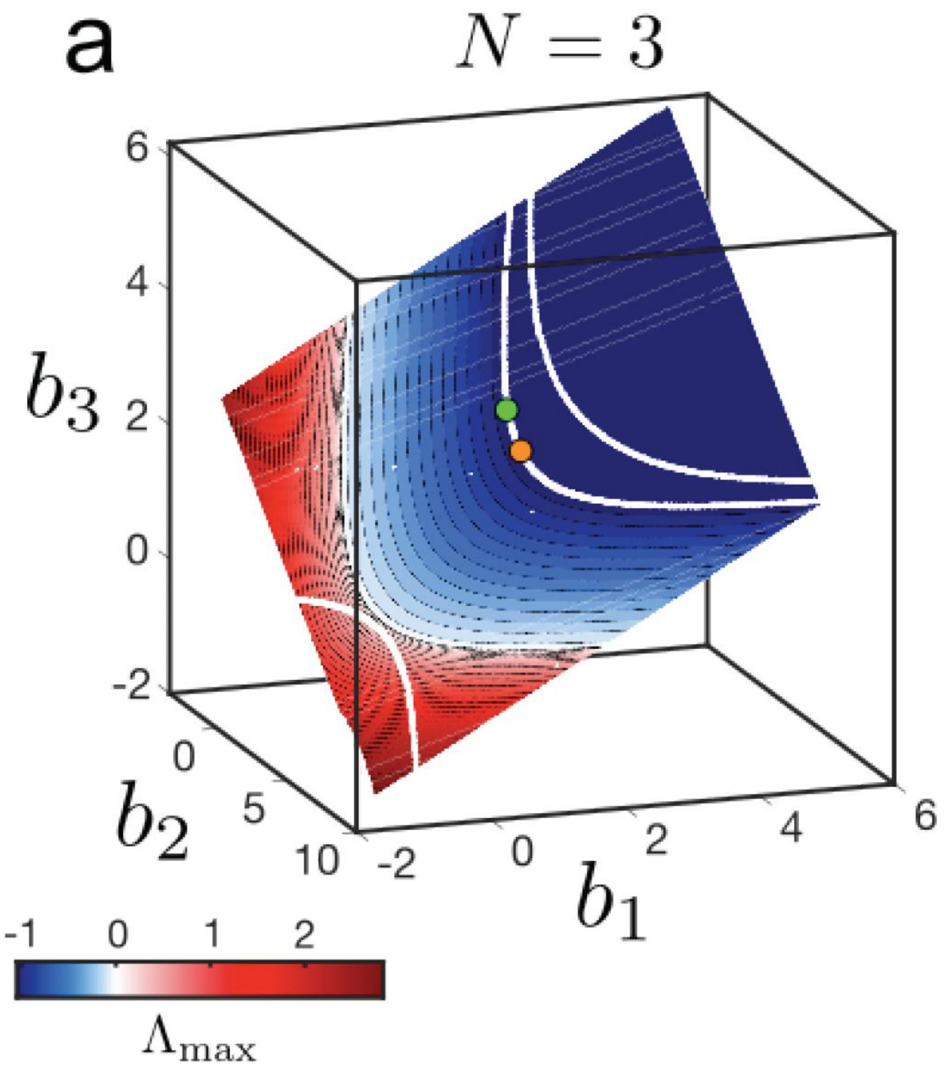
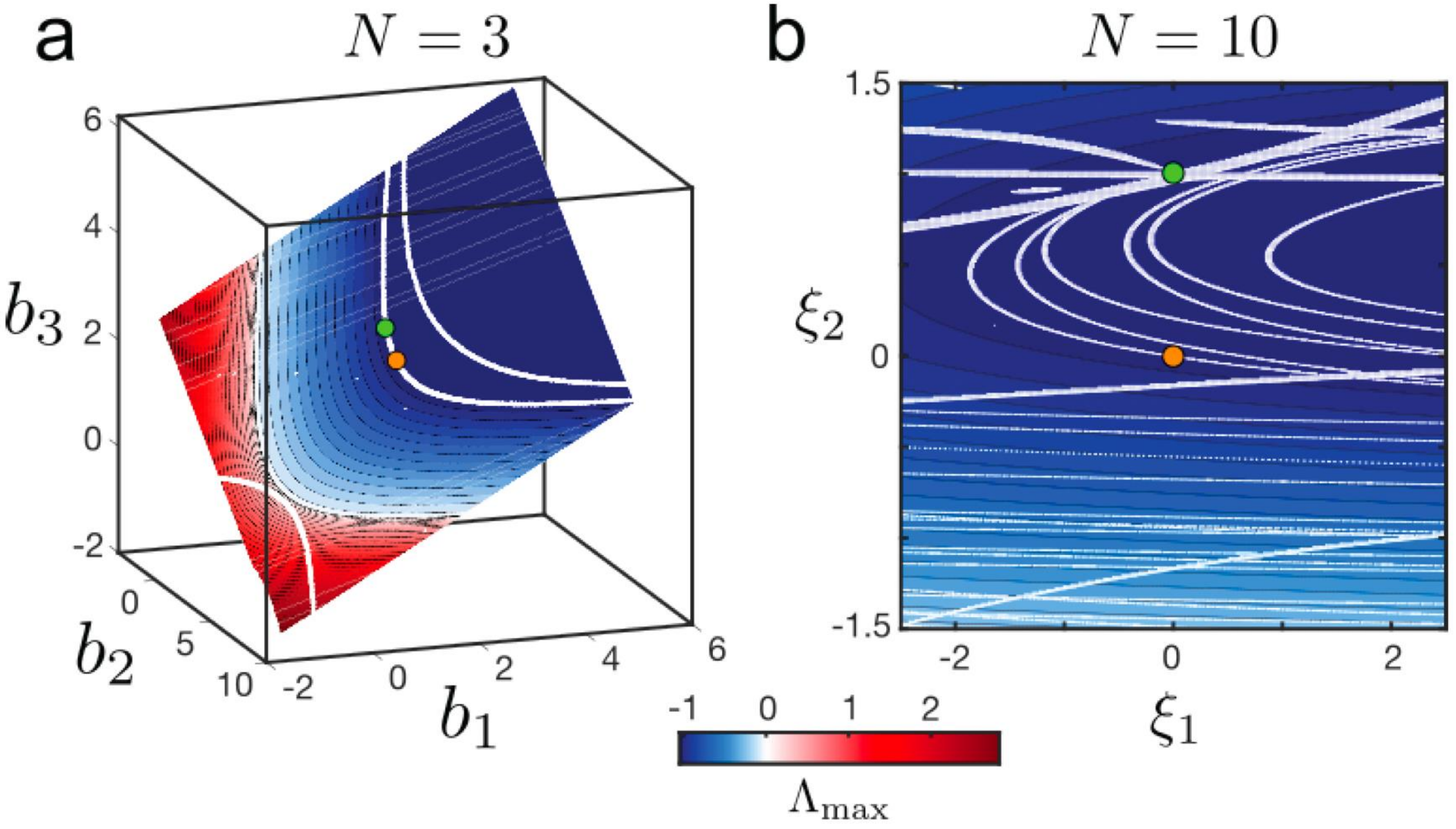


Why? An eigenvalue problem



Questions

Which conditions lead to a heterogeneous optimum?

How to efficiently locate them (algorithmically)?

What are the implications for other network dynamics?

Special thanks

Center for Network Dynamics

Optimal flock formation induced by heterogeneity

Arthur N. Montanari[a,b,1,2], Ana Elisa D. Barioni[a,b,1], Chao Duan[c], and Adilson E. Motter[a,b,d,e]

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[b]Center for Network Dynamics, Northwestern University, Evanston, IL 60208, USA
[c]School of Electrical Engineering, Xi'an Jiaotong University, Xi'an 710049, China
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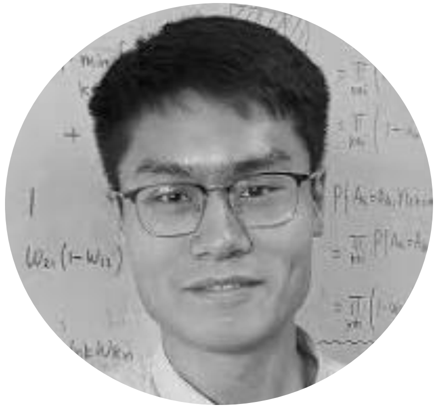
[e]Northwestern Institute on Complex Systems, Northwestern University, Evanston, IL 60208, USA

Ana Barioni



Chao Duan

Adilson Motter

Camila Felix

Code available at https://github.com/montanariarthur/OptFlock

Final remarks

The physics and diversity behind flocking: a nature-inspired study

Network Science Society

Thank you! ... Questions?