SIAM Conference on Applications of Dynamical Systems 2025

Flocking dynamics promoted by heterogeneity

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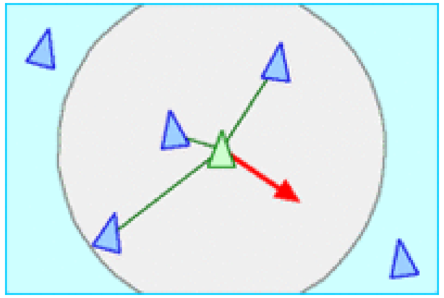
NATIONAL GEOGRAPHIC

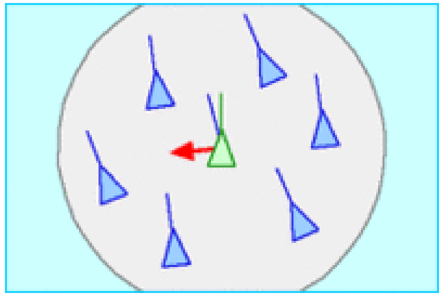
We know a lot of factual information about the starling—its size and voice, where it lives, how it breeds and migrates—but what remains a mystery is how it flies in murmurations, or flocks, without colliding.

Reynold's rules for flocking

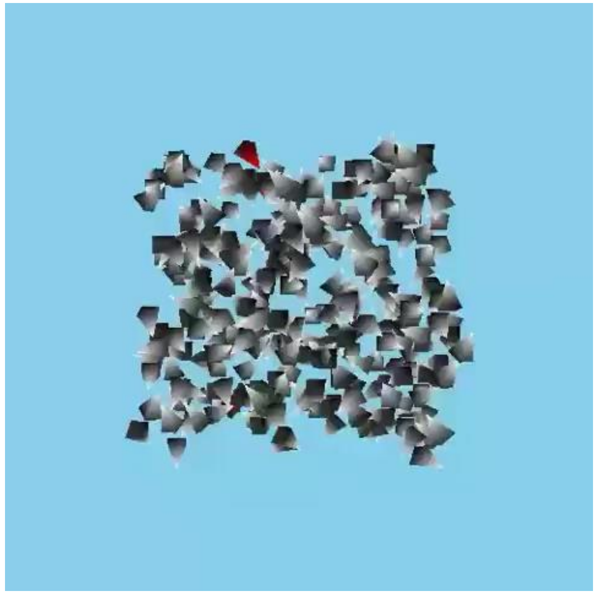
agents must:

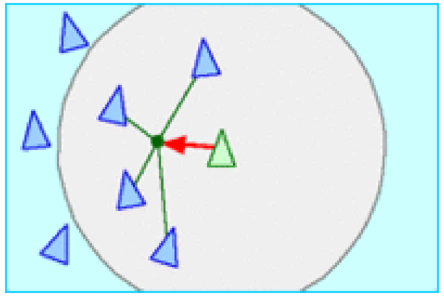


1) avoid collisions with each other (separation)



2) match their velocity to nearby mates (alignment)



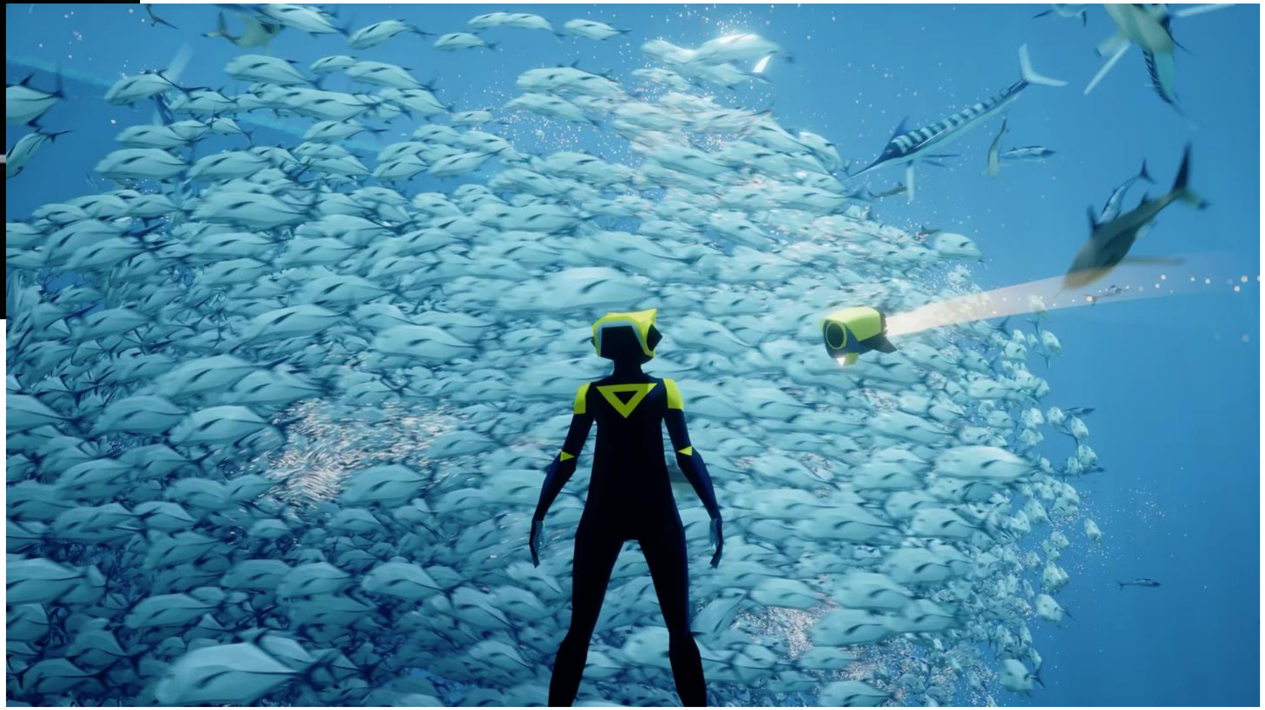


3) move towards the center of mass of its neighbors (cohesion)

C Reynolds. Flocks, herds and schools: A distributed behavioral model. *Computer Graphics* 21 (1985).

Boids: state of the art

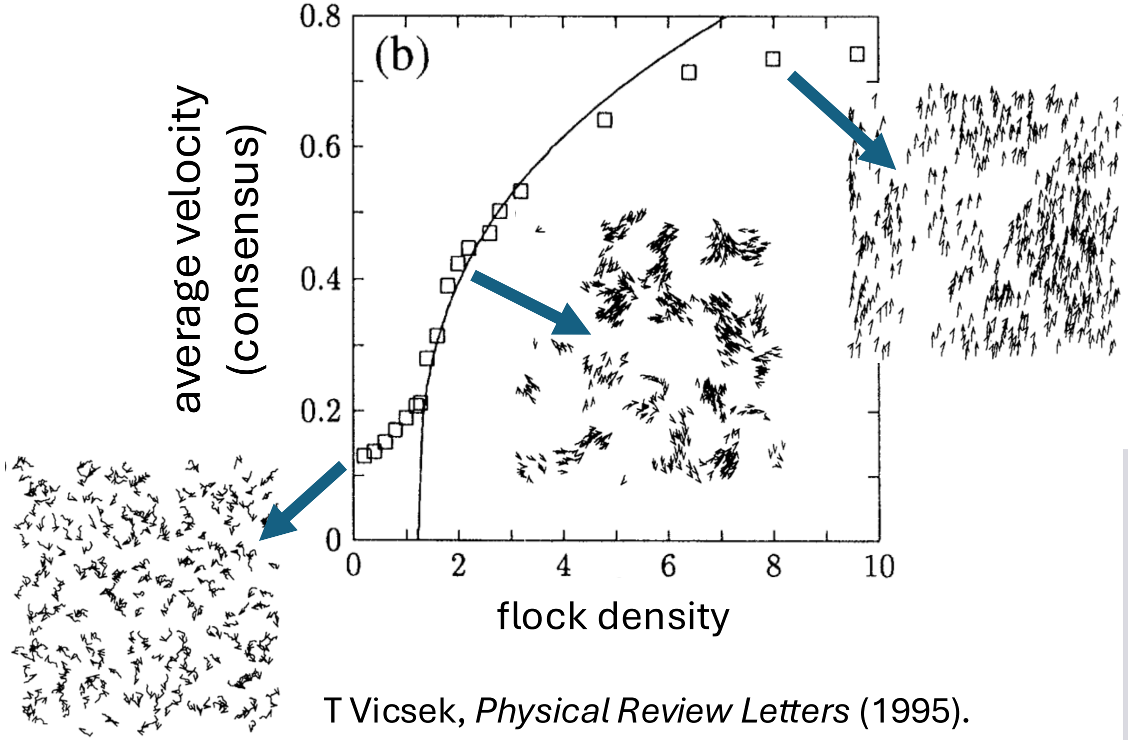
Batman Returns
1992



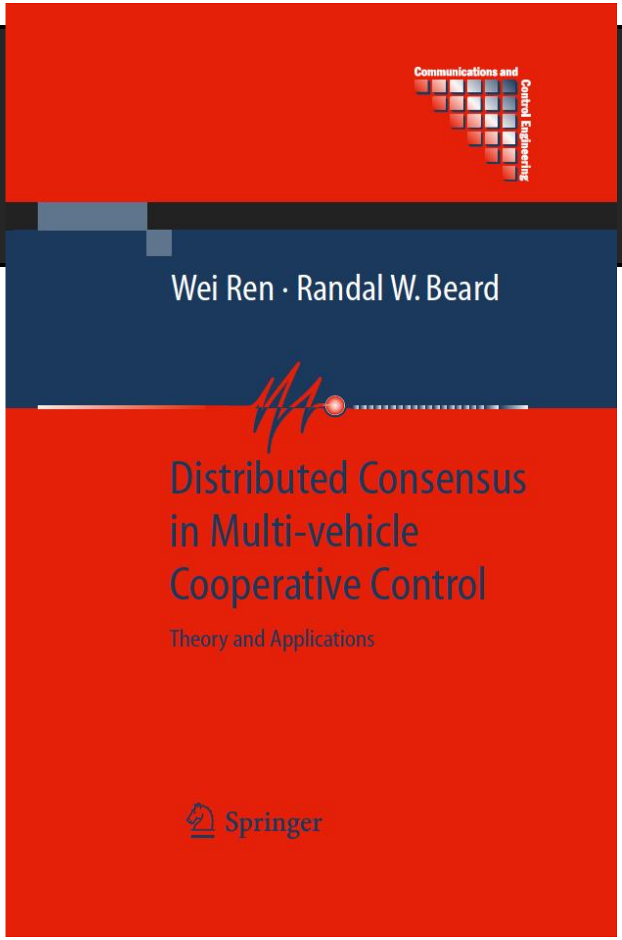
Abzu
2016

The many communities

$$\dot{x}_i = f(x_i) + \text{interaction network}$$



T Vicsek, *Physical Review Letters* (1995).



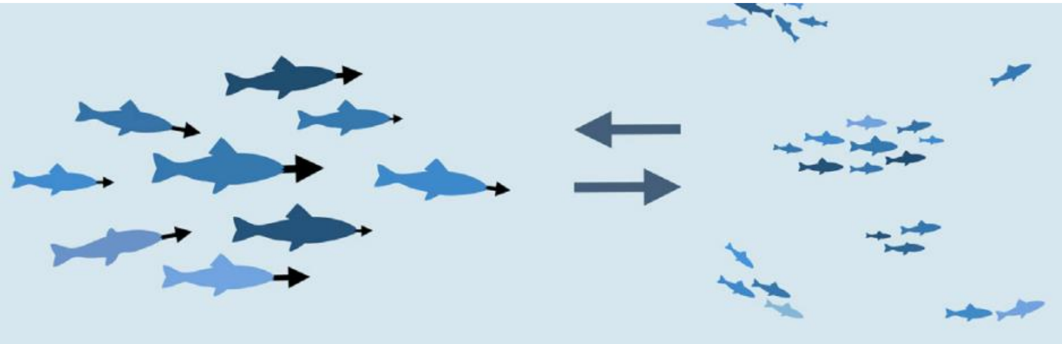
Revealing the hidden networks of interaction in mobile animal groups allows prediction of complex behavioral contagion

Sara Brin Rosenthal[a,1], Colin R. Twomey[b,1], Andrew T. Hartnett[a], Hai Shan Wu[b], and Iain D. Couzin[b,c,d,2]

Departments of [a]Physics and [b]Ecology and Evolutionary Biology, Princeton University, Princeton, NJ 08544; [c]Department of Collective Behaviour, Max Planck Institute for Ornithology, D-78547 Konstanz, Germany; and [d]Chair of Biodiversity and Collective Behavior, Department of Biology, University of Konstanz, D-78547 Konstanz, Germany

Edited by Gene E. Robinson, University of Illinois at Urbana–Champaign, Urbana, IL, and approved February 24, 2015 (received for review October 22, 2014)

On agent heterogeneity

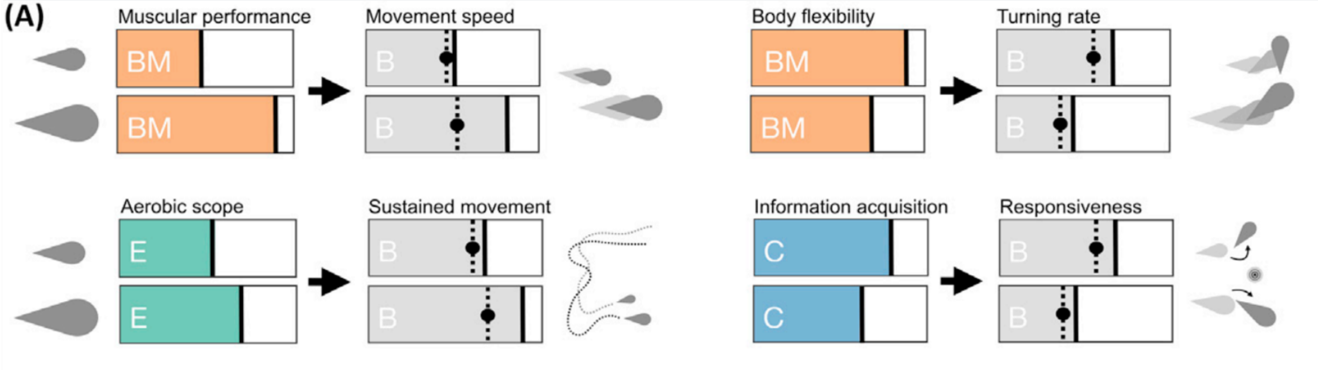


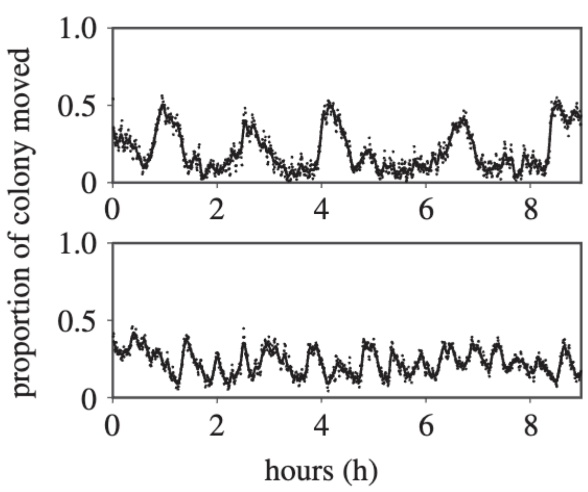
CellPress REVIEWS

Review

The Role of Individual Heterogeneity in Collective Animal Behaviour

Jolle W. Jolles,[1,2,3,7,@,*] Andrew J. King,[4,5] and Shaun S. Killen[6]





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Research

Check for updates

Cite this article: Doering GN, Drawert B, Lee C, Pruitt JN, Petzold LR, Dalnoki-Veress K. 2022 Noise resistant synchronization and collective rhythm switching in a model of animal group locomotion. *R. Soc. Open Sci.* **9**: 211908. https://doi.org/10.1098/rsos.211908

Noise resistant synchronization and collective rhythm switching in a model of animal group locomotion

Grant Navid Doering[1], Brian Drawert[3], Carmen Lee[2], Jonathan N. Pruitt[1], Linda R. Petzold[4,5] and Kari Dalnoki-Veress[2]

Yet, in the Engineering community...

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A tool for analysis and synthesis of heterogeneous multi-agent systems under rank-deficient coupling☆

Jin Gyu Lee [a], Hyungbo Shim [b,*]

[a] Control Group, Department of Engineering, University of Cambridge, Cambridge, United Kingdom
[b] ASRI, Department of Electrical and Computer Engineering, Seoul National University, Seoul, Republic of Korea

Consensus of heterogeneous multi-agent systems

Y. Zheng[1] Y. Zhu[1] L. Wang[2]

[1]Center for Complex Systems, School of Mechano-electronic Engineering, Xidian University, Xi'an 710071, People's Republic of China
[2]Center for Systems and Control, College of Engineering and Key Laboratory of Machine Perception (Ministry of Education), Peking University, Beijing 100871, People's Republic of China
E-mail: zhengyuanshi2005@163.com

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ELSEVIER

Cooperative control of heterogeneous multi-agent systems under spatiotemporal constraints☆

Fei Chen *, Mayank Sewlia, Dimos V. Dimarogonas

Division of Decision and Control Systems, KTH Royal Institute of Technology, SE-100 44, Stockholm, Sweden

> Small mismatches
> Random parameter mismatches

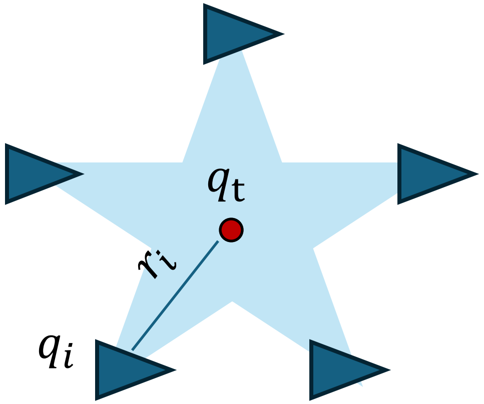
Take-home message

Consensus can be achieved and enhanced not despite, but **because of heterogeneity**.

Flocking model for target tracking

$$\dot{\boldsymbol{q}}_i = \boldsymbol{p}_i,$$

$$m_i \dot{\boldsymbol{p}}_i = \dot{\boldsymbol{p}}_{\mathrm{t}} - b_i\left(\boldsymbol{q}_i - \boldsymbol{q}_{\mathrm{t}} - \boldsymbol{r}_i\right) - \gamma c_i\left(\boldsymbol{p}_i - \boldsymbol{p}_{\mathrm{t}}\right) + \sum_{j=1}^{N} A_{ij}(t)\left[(\boldsymbol{q}_j - \boldsymbol{r}_j) - (\boldsymbol{q}_i - \boldsymbol{r}_i) + \gamma(\boldsymbol{p}_j - \boldsymbol{p}_i)\right]$$

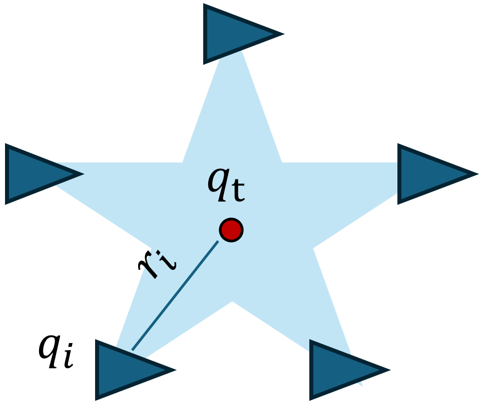


pre-specified formation: $\boldsymbol{q}_i(t) \to \boldsymbol{q}_{\mathrm{t}}(t) + \boldsymbol{r}_i$

trajectory tracking: $\boldsymbol{p}_i(t) \to \boldsymbol{p}_{\mathrm{t}}(t)$ as $t \to \infty$

Flocking model for target tracking

$$\dot{\boldsymbol{q}}_i = \boldsymbol{p}_i,$$
$$m_i \dot{\boldsymbol{p}}_i = \dot{\boldsymbol{p}}_{\mathrm{t}} - b_i \left(\boldsymbol{q}_i - \boldsymbol{q}_{\mathrm{t}} - \boldsymbol{r}_i\right) - \gamma c_i \left(\boldsymbol{p}_i - \boldsymbol{p}_{\mathrm{t}}\right)$$
$$+ \sum_{j=1}^{N} A_{ij}(t) \Big[(\boldsymbol{q}_j - \boldsymbol{r}_j) - (\boldsymbol{q}_i - \boldsymbol{r}_i) + \gamma(\boldsymbol{p}_j - \boldsymbol{p}_i) \Big]$$

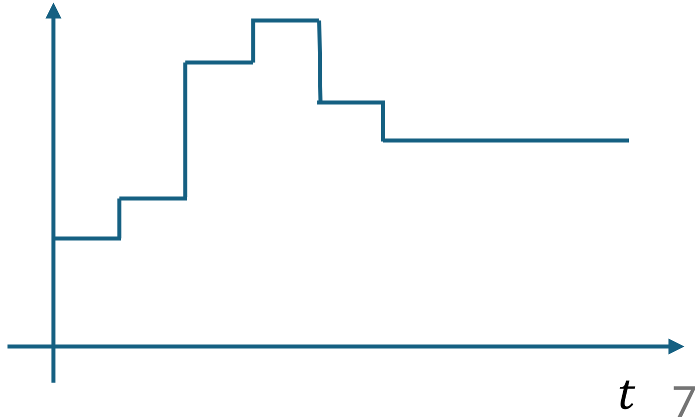


pre-specified formation: $\boldsymbol{q}_i(t) \to \boldsymbol{q}_{\mathrm{t}}(t) + \boldsymbol{r}_i$

trajectory tracking: $\boldsymbol{p}_i(t) \to \boldsymbol{p}_{\mathrm{t}}(t)$ as $t \to \infty$

adjacency matrix: $\tilde{A}_{ij}(t) = K/(\rho^2 + \|\boldsymbol{q}_i(t) - \boldsymbol{q}_j(t)\|^2)^\beta$

all-to-all, weighted,
time dependent,
piecewise constant



Flocking model for target tracking

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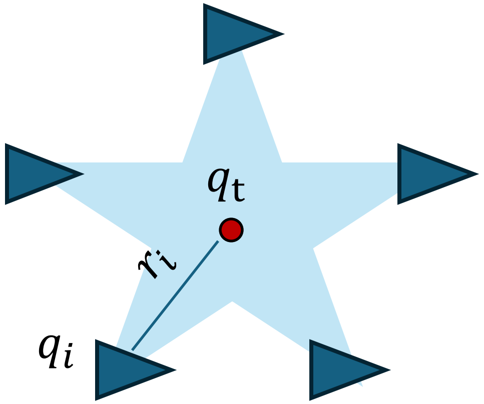
$$m_i \dot{\boldsymbol{p}}_i = \dot{\boldsymbol{p}}_{\mathrm{t}} - b_i \left(\boldsymbol{q}_i - \boldsymbol{q}_{\mathrm{t}} - \boldsymbol{r}_i\right) - \gamma c_i \left(\boldsymbol{p}_i - \boldsymbol{p}_{\mathrm{t}}\right)$$
$$+ \sum_{j=1}^{N} A_{ij}(t) \Big[(\boldsymbol{q}_j - \boldsymbol{r}_j) - (\boldsymbol{q}_i - \boldsymbol{r}_i) + \gamma (\boldsymbol{p}_j - \boldsymbol{p}_i) \Big]$$



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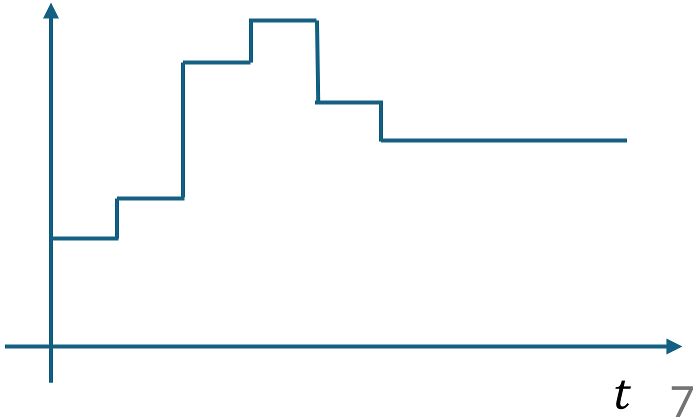
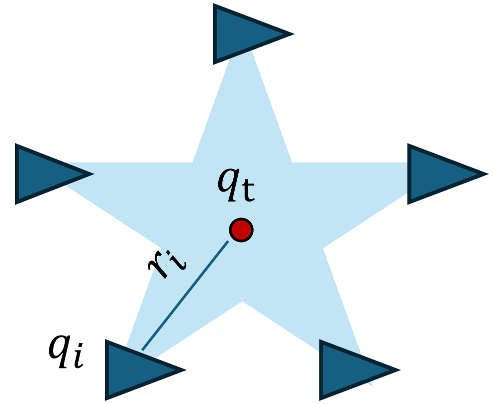


Dynamics of the tracking error

LTV system $[\boldsymbol{e}_{q,i},\ \boldsymbol{e}_{p,i}] = [\boldsymbol{q}_i - (\boldsymbol{q}_{\mathrm{t}} + \boldsymbol{r}_i),\ \boldsymbol{p}_i - \boldsymbol{p}_{\mathrm{t}}]$



$$\begin{bmatrix} \dot{\boldsymbol{e}}_q \\ \dot{\boldsymbol{e}}_p \end{bmatrix} = \underbrace{\begin{bmatrix} 0_{Nm} & I_{Nm} \\ -I_m \otimes M(B+L(t)) & -I_m \otimes \gamma M(C+L(t)) \end{bmatrix}}_{J(t)} \underbrace{\begin{bmatrix} \boldsymbol{e}_q \\ \boldsymbol{e}_p \end{bmatrix}}_{\boldsymbol{e}(t)}$$

$$\begin{bmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_N \end{bmatrix}$$

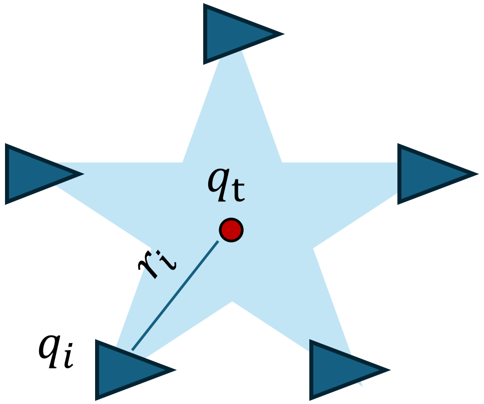
$$\begin{bmatrix} m_1^{-1} & & & \\ & m_2^{-1} & & \\ & & \ddots & \\ & & & m_N^{-1} \end{bmatrix} \qquad \begin{bmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_N \end{bmatrix}$$

Laplacian matrix

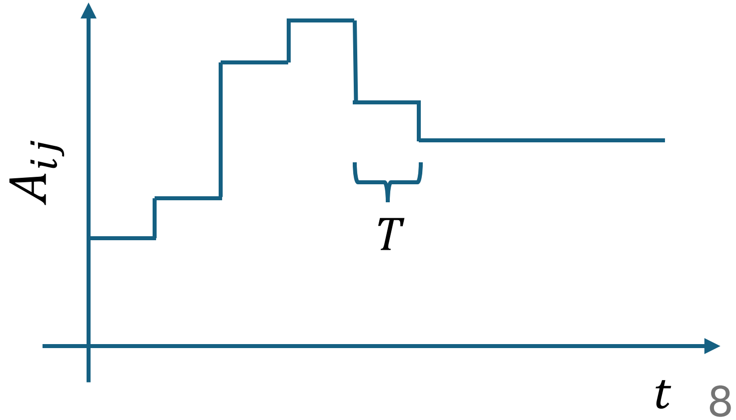
Dynamics of the tracking error

LTV system $\quad [\boldsymbol{e}_{q,i},\ \boldsymbol{e}_{p,i}] = [\boldsymbol{q}_i - (\boldsymbol{q}_\mathrm{t} + \boldsymbol{r}_i),\ \boldsymbol{p}_i - \boldsymbol{p}_\mathrm{t}]$

$$\begin{bmatrix} \dot{\boldsymbol{e}}_q \\ \dot{\boldsymbol{e}}_p \end{bmatrix} = \underbrace{\begin{bmatrix} 0_{Nm} & I_{Nm} \\ -I_m \otimes M(B+L(t)) & -I_m \otimes \gamma M(C+L(t)) \end{bmatrix}}_{J(t)} \underbrace{\begin{bmatrix} \boldsymbol{e}_q \\ \boldsymbol{e}_p \end{bmatrix}}_{\boldsymbol{e}(t)}$$



$$\|\boldsymbol{e}(t)\| \le \eta \exp\left\{\sum_{k=0}^{t/T} \Lambda_{\max}(J(t_k))T\right\}\|\boldsymbol{e}(0)\|$$



Optimal flocking dynamics

$$\|\boldsymbol{e}(t)\| \leq \eta \exp\left\{\sum_{k=0}^{t/T} \Lambda_{\max}(J(t_k))T\right\}\|\boldsymbol{e}(0)\|$$

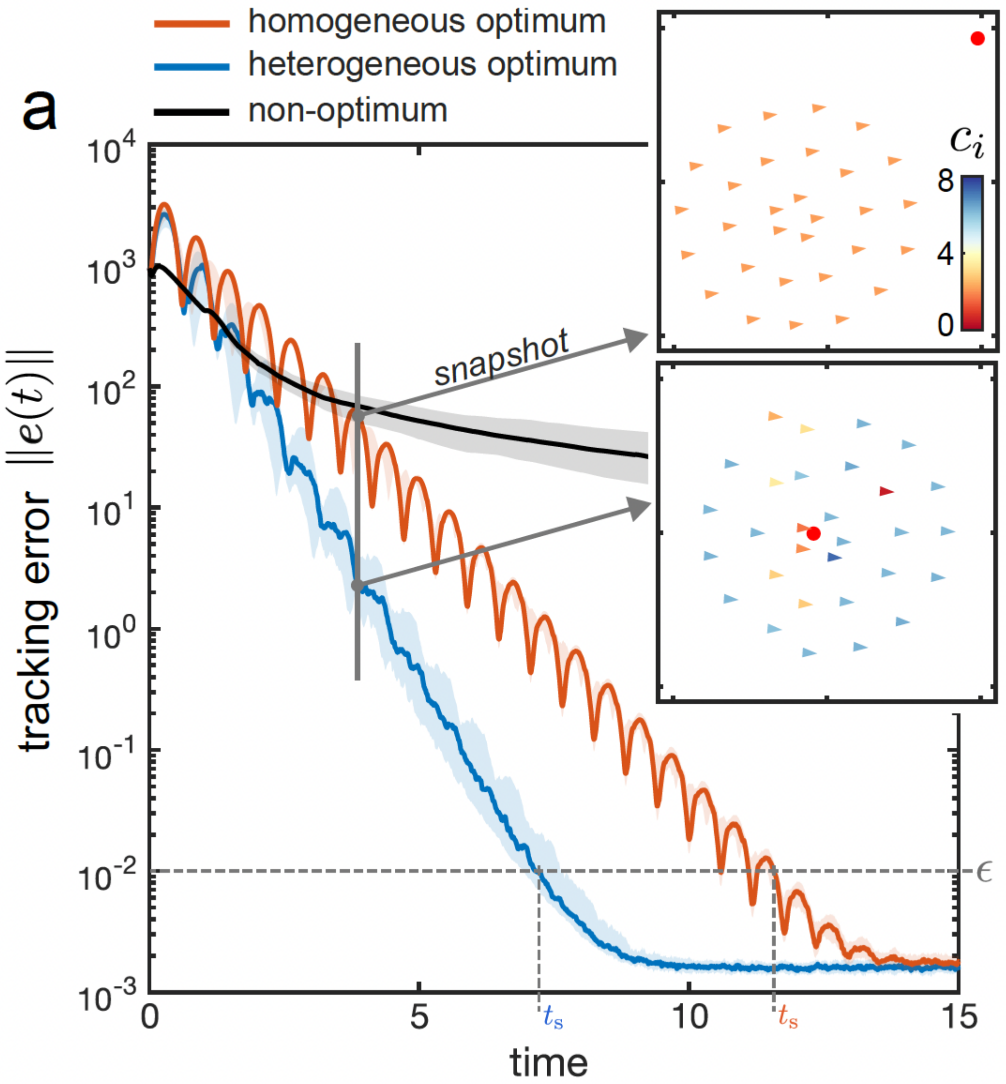
<u>Optimal control procedure</u>

$$\begin{aligned} \min_{\boldsymbol{b},\boldsymbol{c}} \quad & \Lambda_{\max}(J(t_k)), \\ \text{s.t.} \quad & 0 < \boldsymbol{b} \leq b_{\max}, \\ & 0 < \boldsymbol{c} \leq c_{\max}, \end{aligned}$$

solved in ”real time” at each time interval $[t_k, t_k + T]$

1. optimal flocks of *homogeneous* agents, where parameters are optimized subject to the constraint that all agents have identical gains, i.e., $\boldsymbol{b}^{(k)} = [b^{(k)}, \ldots, b^{(k)}]$ and $\boldsymbol{c}^{(k)} = [c^{(k)}, \ldots, c^{(k)}]$;
2. optimal flocks of *heterogeneous* agents, where gains are optimized independently for each agent, i.e., $\boldsymbol{b}^{(k)} = [b_1^{(k)}, \ldots, b_N^{(k)}]$ and $\boldsymbol{c}^{(k)} = [c_1^{(k)}, \ldots, c_N^{(k)}]$.

Heterogeneous vs homogeneous flocking

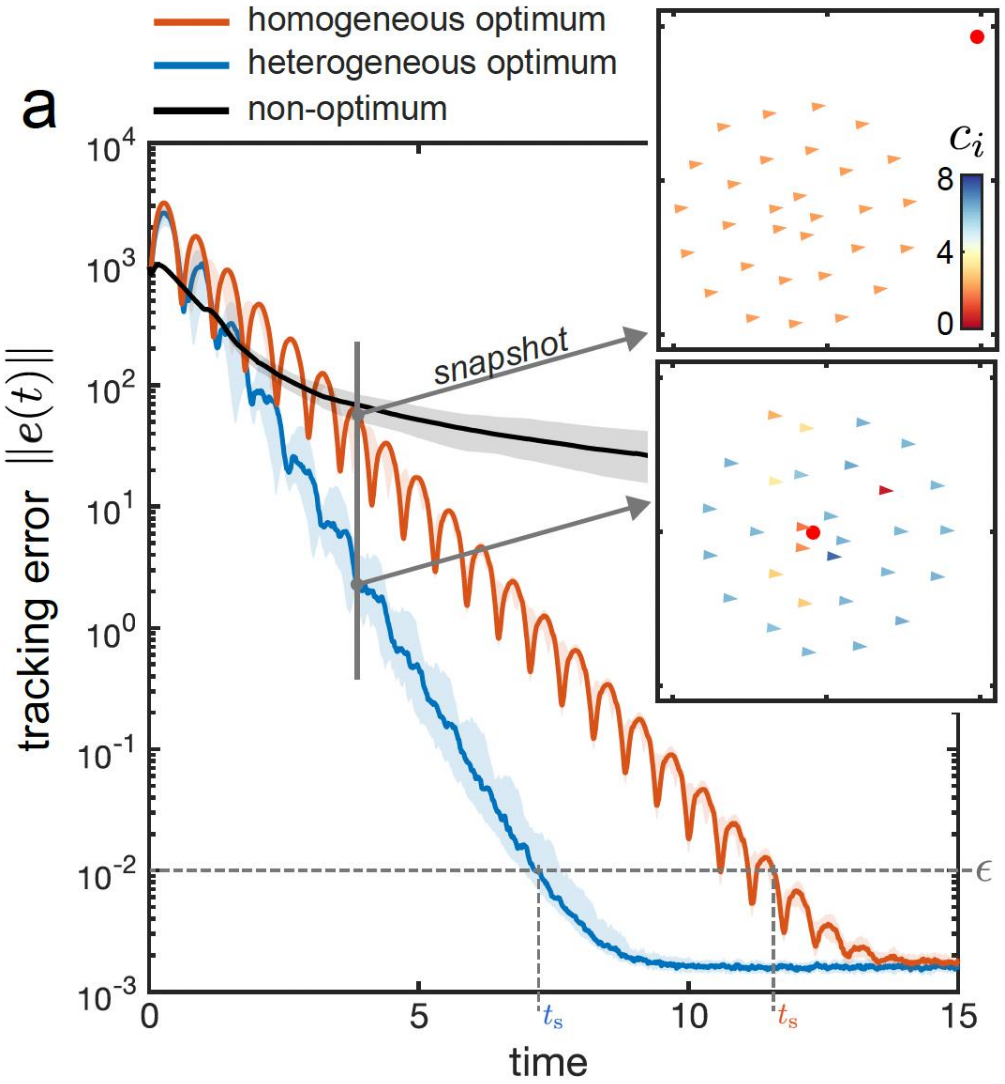


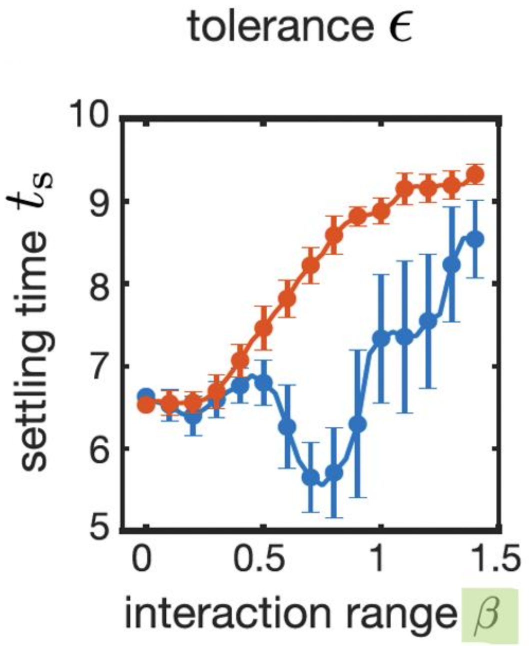
36% improvement

Supplementary Movie 1

Target Tracking and Flock Formation

Heterogeneous vs homogeneous flocking





$$\tilde{A}_{ij}(t) = K/(\rho^2 + \|\boldsymbol{q}_i(t) - \boldsymbol{q}_j(t)\|^2)^{\beta}$$

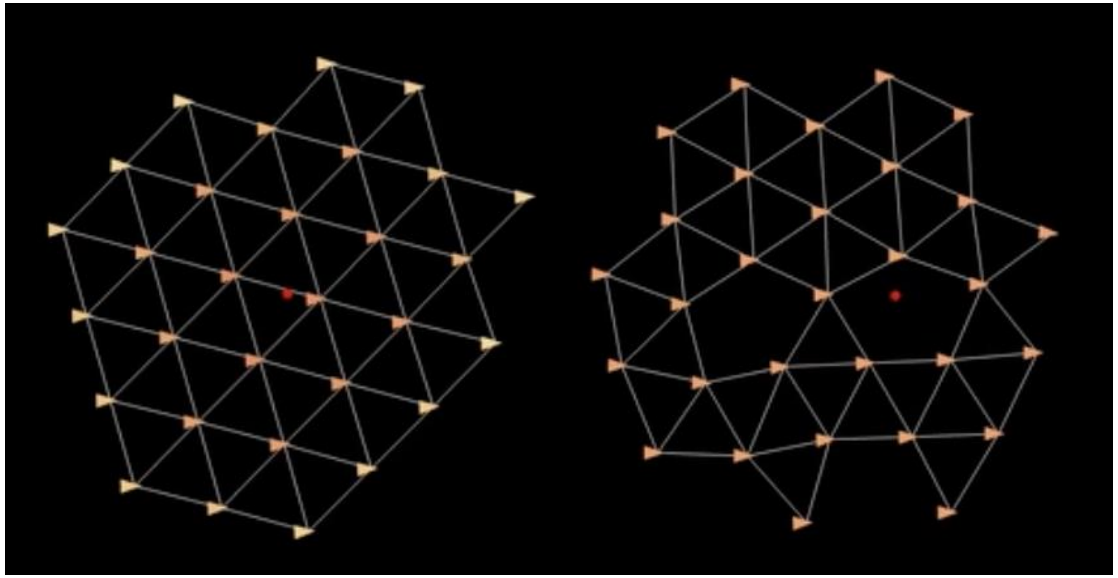
$$B(t) + L(t) = \begin{bmatrix} b_1 + \sum_j A_{1j} & -A_{12} & \dots & -A_{1N} \\ -A_{21} & b_2 + \sum_j A_{2j} & \dots & -A_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -A_{N1} & -A_{N2} & \dots & b_N + \sum_j A_{Nj} \end{bmatrix}$$

Extension to free-flocking model

$$\dot{\boldsymbol{q}}_i = \boldsymbol{p}_i ,$$

$$\dot{\boldsymbol{p}}_i = \boldsymbol{u}_i^{\alpha} + \boldsymbol{u}_i^{\gamma} + \boldsymbol{u}_i^{\beta} ,$$

R Olfati-Saber, *IEEE Trans. Automatic Control* (2006).



model complexity

previous | **current**

Extension to free-flocking model

$$\dot{\boldsymbol{q}}_i = \boldsymbol{p}_i,$$
$$\dot{\boldsymbol{p}}_i = \boldsymbol{u}_i^\alpha + \boldsymbol{u}_i^\gamma + \boldsymbol{u}_i^\beta,$$

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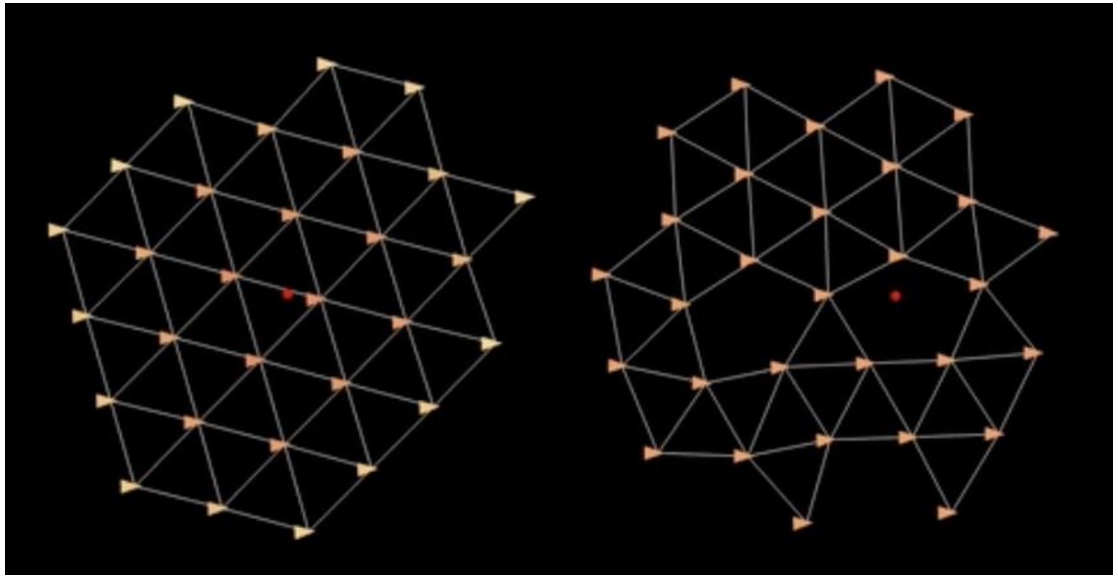

agent-agent interaction:

$$\boldsymbol{u}_i^\alpha = -k_1^\alpha \underbrace{\nabla_{q_i} V(q)}_{\substack{V(\boldsymbol{q}^*) = 0 \\ \text{iff } \|\boldsymbol{q}_i^* - \boldsymbol{q}_j^*\| = d}} + k_2^\alpha \sum_{j \in \mathcal{N}_i(\boldsymbol{q})} A_{ij}(\boldsymbol{q})(\boldsymbol{p}_j - \boldsymbol{p}_i)$$

model complexity

previous	current
pre-assigned formation	emergent formation

Extension to free-flocking model

$$\dot{\boldsymbol{q}}_i = \boldsymbol{p}_i\,,$$

$$\dot{\boldsymbol{p}}_i = \boldsymbol{u}_i^\alpha + \boldsymbol{u}_i^\gamma + \boldsymbol{u}_i^\beta\,,$$

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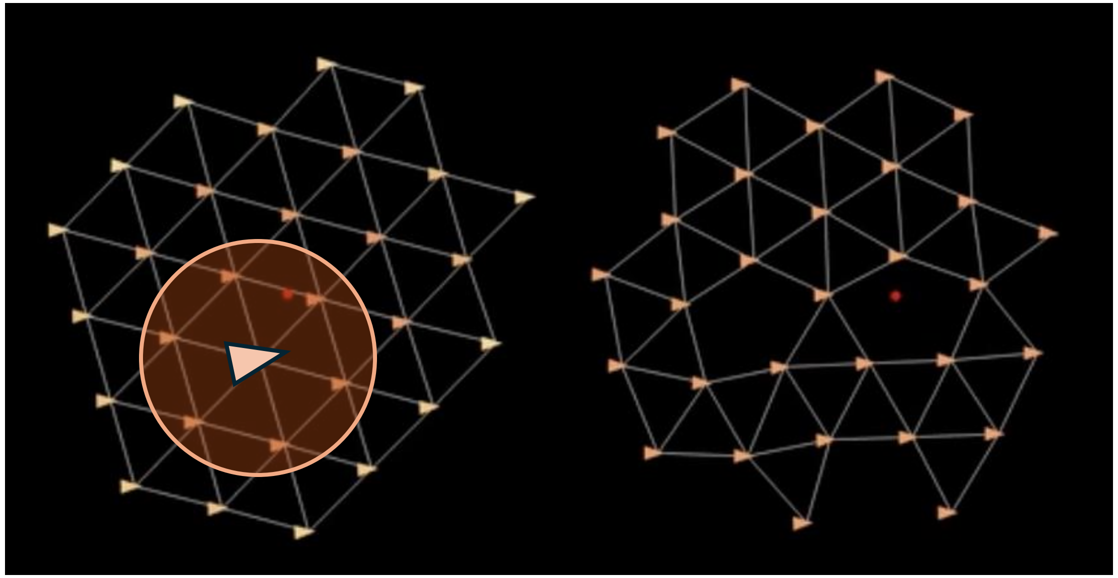

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model complexity

previous	**current**
pre-assigned formation	emergent formation
all-to-all, weighted network	sparse, weighted network
piecewise constant adj. matrix	continuous adjacency matrix
linear time-varying dynamics	nonlinear dynamics

Extension to free-flocking model

$$\dot{\boldsymbol{q}}_i = \boldsymbol{p}_i\,,$$

$$\dot{\boldsymbol{p}}_i = \boldsymbol{u}_i^{\alpha} + \boldsymbol{u}_i^{\gamma} + \boldsymbol{u}_i^{\beta}\,,$$

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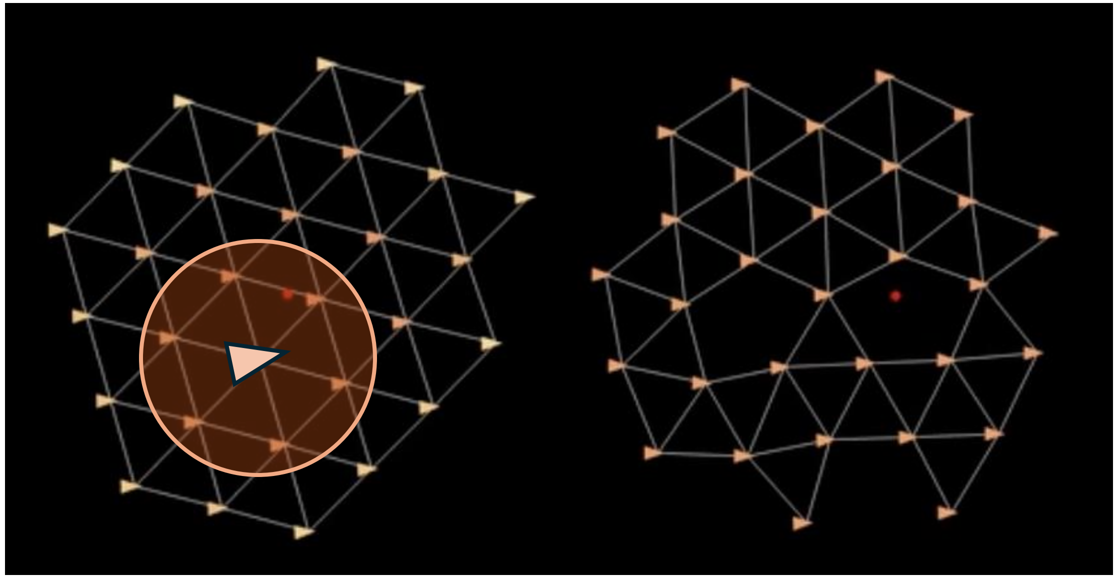


agent-agent interaction:

$$\boldsymbol{u}_i^{\alpha} = -k_1^{\alpha}\boldsymbol{\nabla}_{\boldsymbol{q}_i} V(\boldsymbol{q}) + k_2^{\alpha} \sum_{j \in \mathcal{N}_i(\boldsymbol{q})} A_{ij}(\boldsymbol{q})(\boldsymbol{p}_j - \boldsymbol{p}_i)$$

agent-target interaction:

$$\boldsymbol{u}_i^{\gamma} = -b_i(\boldsymbol{q}_i - \boldsymbol{q}_{\mathrm{t}}) - c_i(\boldsymbol{p}_i - \boldsymbol{p}_{\mathrm{t}})$$

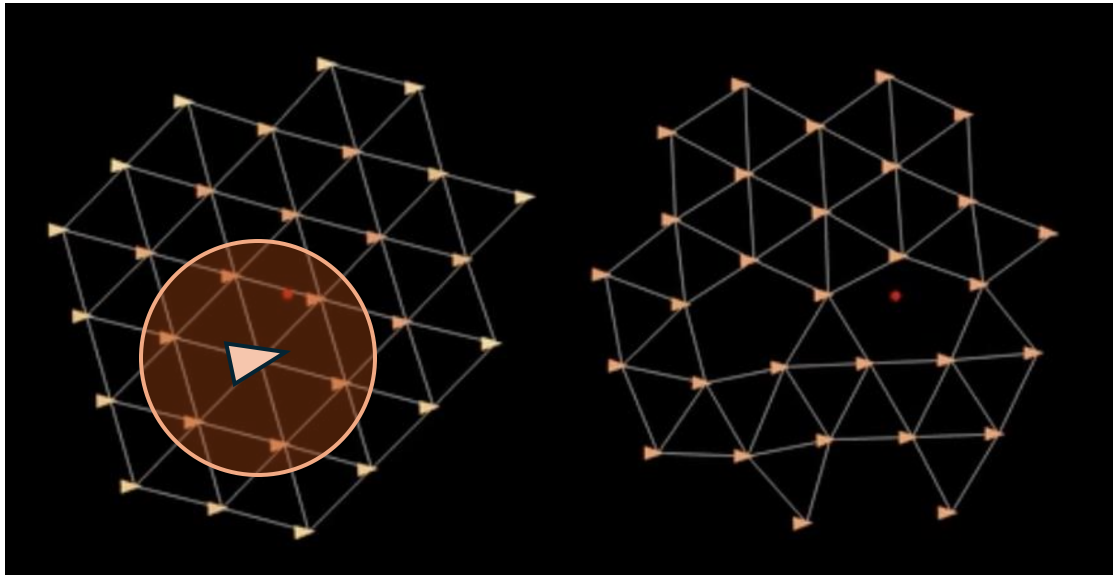
model complexity

previous	**current**
pre-assigned formation	emergent formation
all-to-all, weighted network	sparse, weighted network
piecewise constant adj. matrix	continuous adjacency matrix
linear time-varying dynamics	nonlinear dynamics

Extension to free-flocking model

$$\dot{\boldsymbol{q}}_i = \boldsymbol{p}_i,$$
$$\dot{\boldsymbol{p}}_i = \boldsymbol{u}_i^\alpha + \boldsymbol{u}_i^\gamma + \boldsymbol{u}_i^\beta,$$

R Olfati-Saber, *IEEE Trans. Automatic Control* (2006).



agent-agent interaction:

$$\boldsymbol{u}_i^\alpha = -k_1^\alpha \boldsymbol{\nabla}_{\boldsymbol{q}_i} V(\boldsymbol{q}) + k_2^\alpha \sum_{j \in \mathcal{N}_i(\boldsymbol{q})} A_{ij}(\boldsymbol{q})(\boldsymbol{p}_j - \boldsymbol{p}_i)$$

agent-target interaction:

$$\boldsymbol{u}_i^\gamma = -b_i(\boldsymbol{q}_i - \boldsymbol{q}_{\mathrm{t}}) - c_i(\boldsymbol{p}_i - \boldsymbol{p}_{\mathrm{t}})$$

agent-obstacle interaction:

$\boldsymbol{u}_i^\beta = 0$ (... for now)

model complexity

previous	**current**
pre-assigned formation	emergent formation
all-to-all, weighted network	sparse, weighted network
piecewise constant adj. matrix	continuous adjacency matrix
linear time-varying dynamics	nonlinear dynamics
	3 Reynold's rules

Optimal free flocking

Nonlinear system stability analysis

Lyapunov function: $H(\mathbf{e}) = V(\mathbf{e}_q) + \frac{1}{2}\bar{\mathbf{e}}^{\mathsf{T}}\bar{\mathbf{e}}$

1) $\alpha_1 \|\bar{\mathbf{e}}\|^2 \leq H(\mathbf{e}) \leq \alpha_2 \|\bar{\mathbf{e}}\|^2$

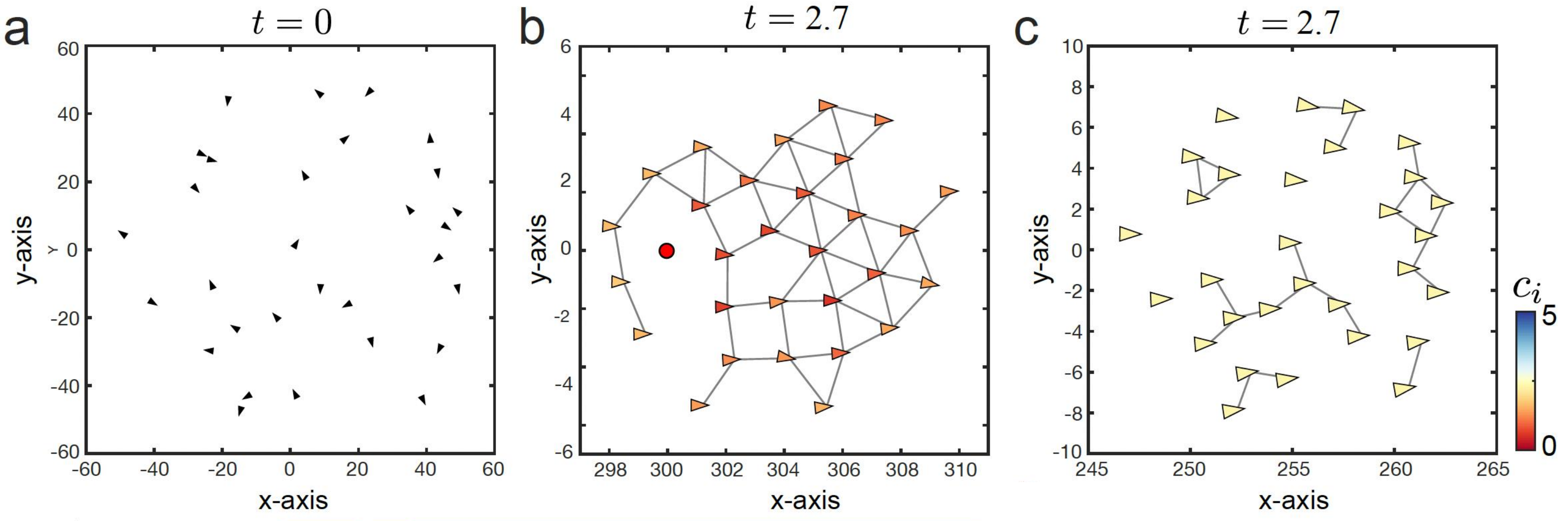
2) $\dot{H}(\mathbf{e}, t) = \bar{\mathbf{e}}^{\mathsf{T}} J(t)\bar{\mathbf{e}} \leq \Lambda_{\max}(J(t_k))\|\bar{\mathbf{e}}\|^2,$ $\quad J(t_k) = \begin{bmatrix} 0_{Nm} & I_{Nm} \\ -I_m \otimes B & -I_m \otimes [C + L(\mathbf{q}(t_k))] \end{bmatrix}$

$$\|\mathbf{e}(t) - \mathbf{e}^*\| \leq \eta \exp\left\{\frac{\eta_k}{2\alpha_2}\Lambda_{\max}(J(t_k))T\right\}\|\mathbf{e}(t_k) - \mathbf{e}^*\|$$

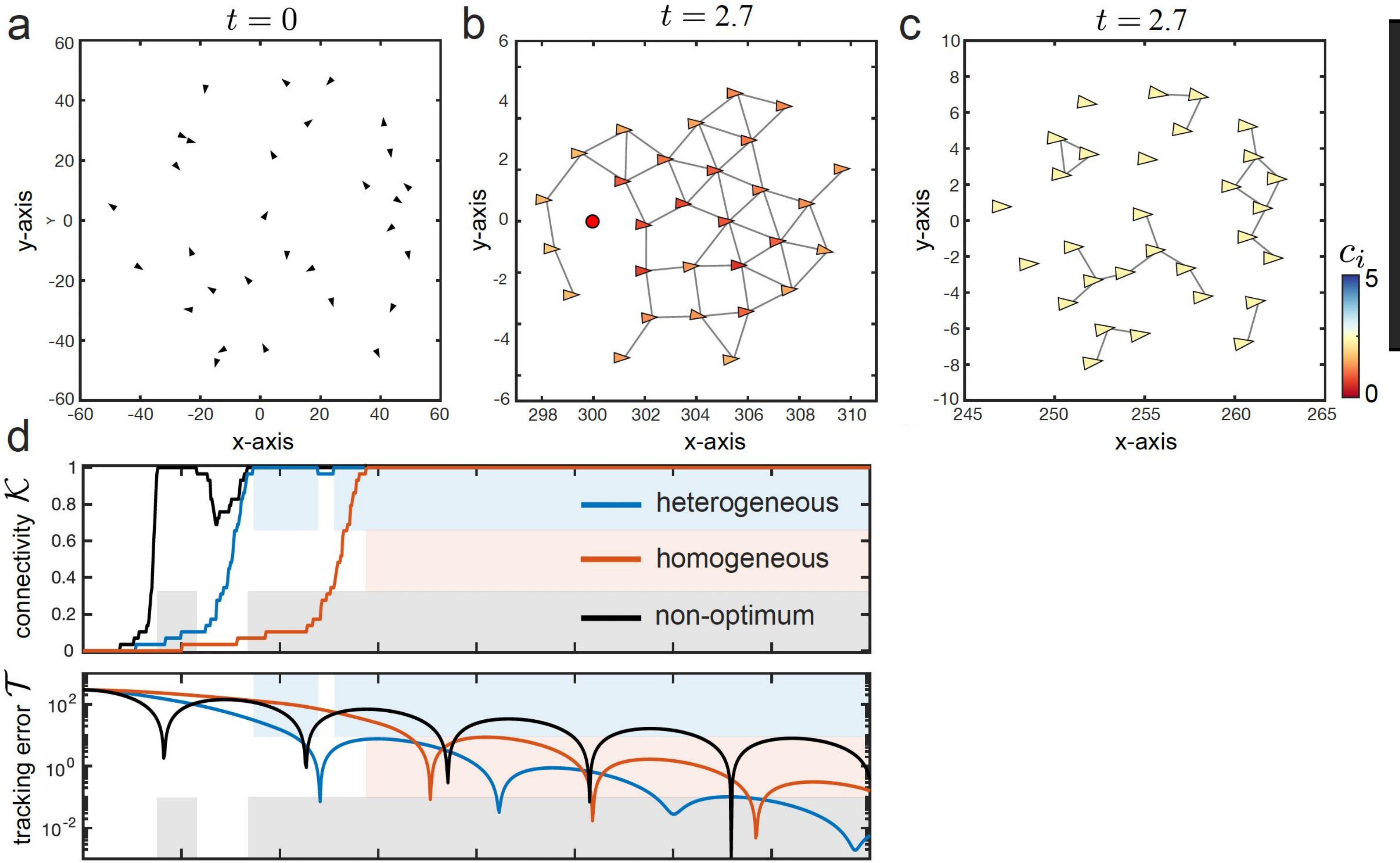
Supplementary Movie 2

Optimal Free Flocking

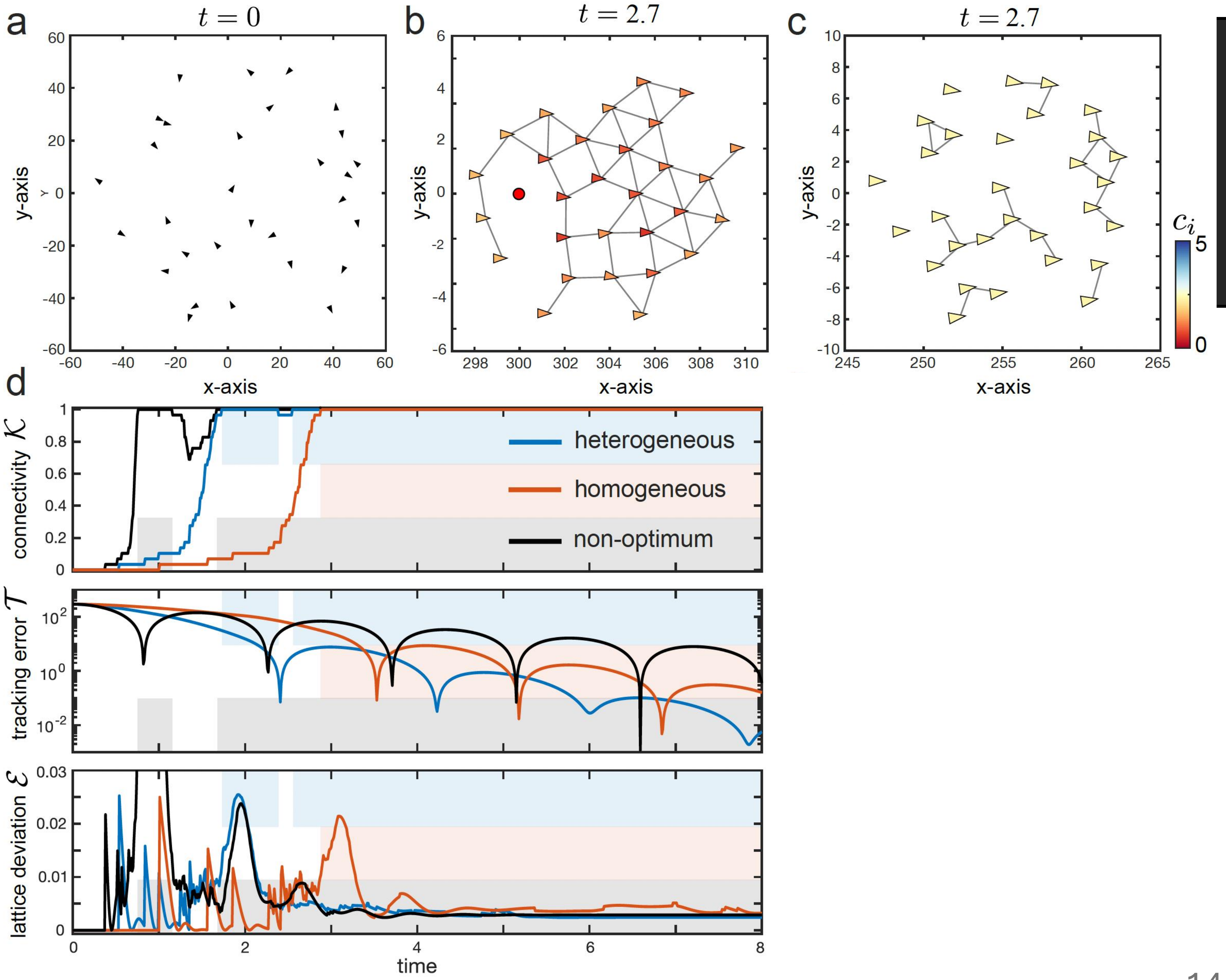
Optimal free flocking



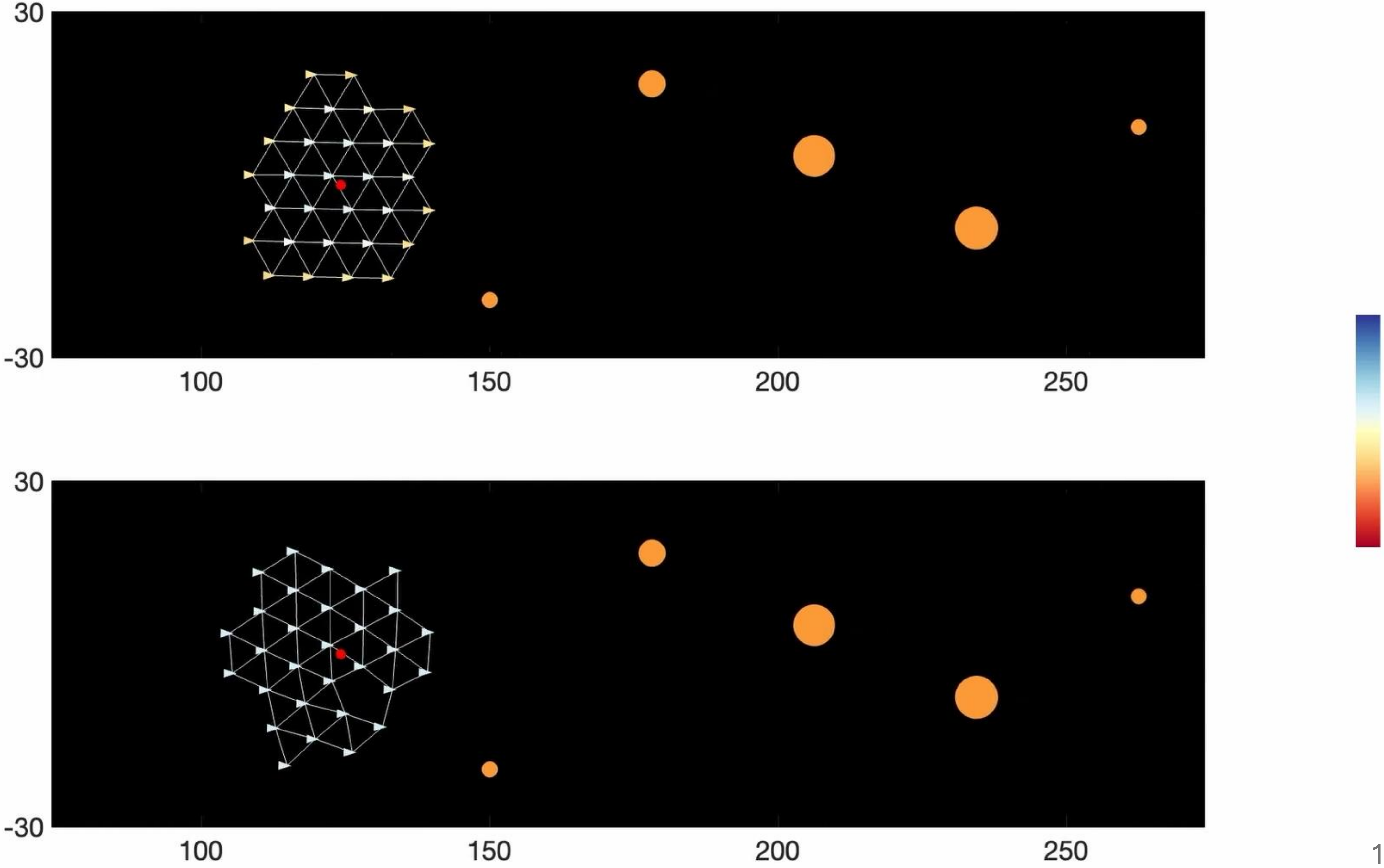
Optimal free flocking



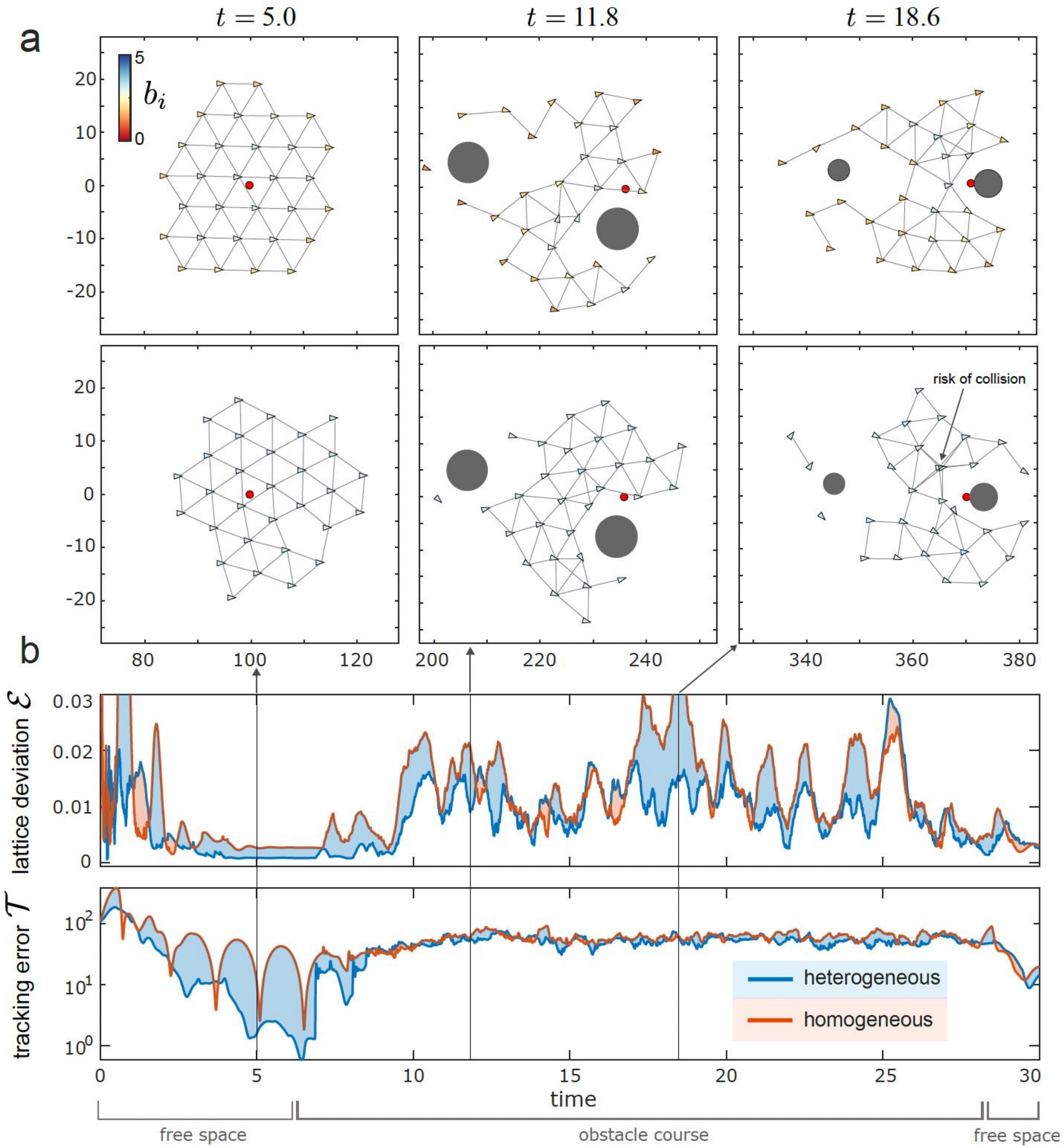
Optimal free flocking



Homogeneous vs. Heterogeneous



Optimal obstacle maneuvering



Take-home question

Why does **heterogeneity** improve **collective** behavior?

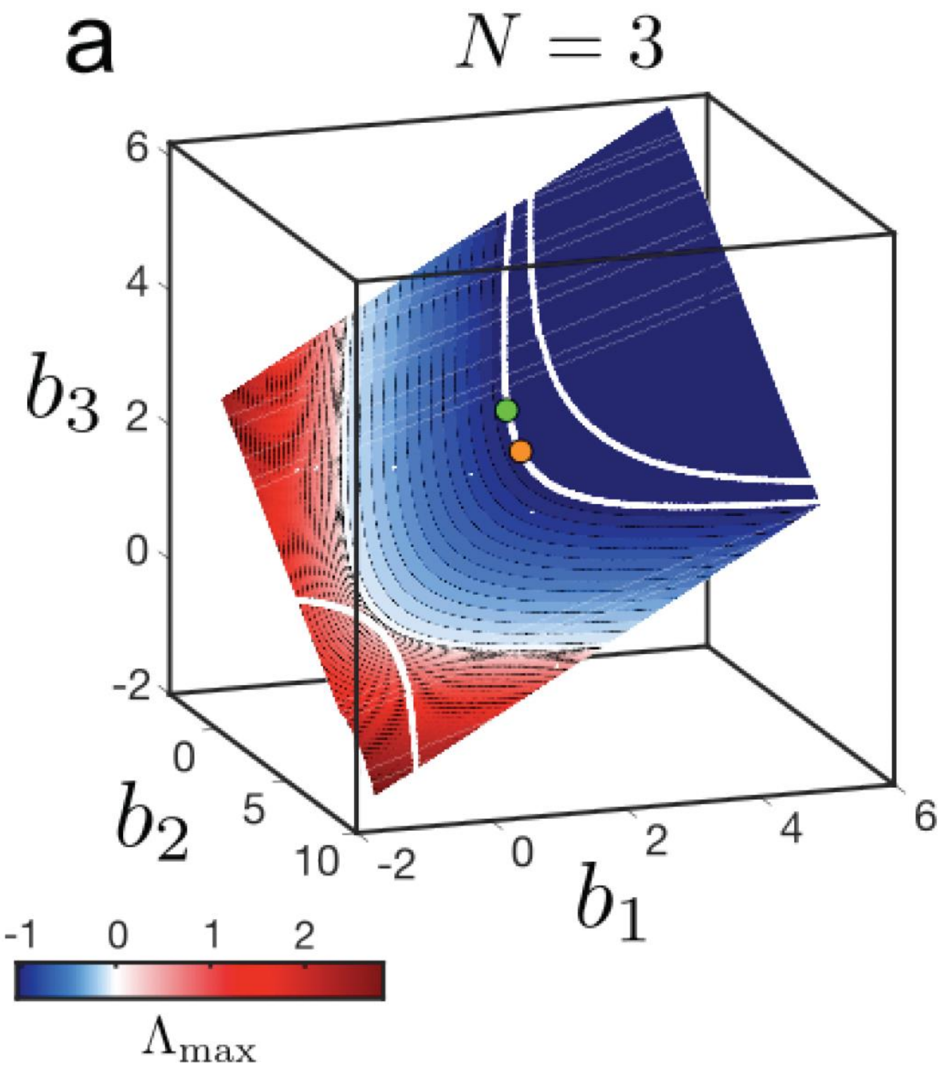
Why? An eigenvalue problem

$$\min_{\boldsymbol{b},\boldsymbol{c}} \quad \Lambda_{\max}(J(t_k)),$$

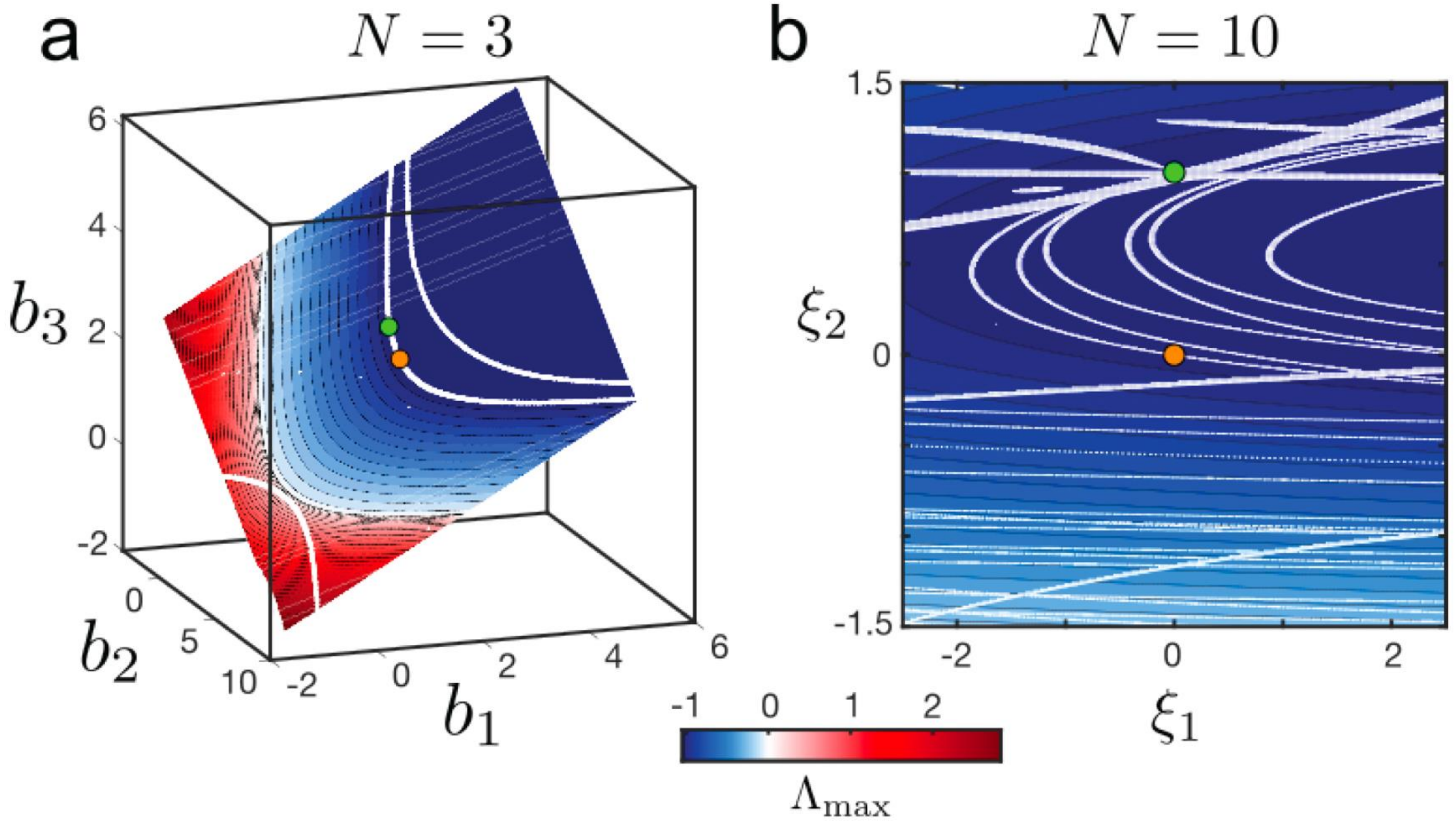
$$\text{s.t.} \quad 0 < \boldsymbol{b} \leq b_{\max},$$

$$J'(\boldsymbol{b}) = \begin{bmatrix} 0_{Nm} & I_{Nm} \\ -(B + L) & -\gamma(B + L) \end{bmatrix}$$

$$b^*_{\text{hom}} = \frac{2}{\gamma^2} - \ell_N + \sqrt{(\ell_N - \ell_1)^2 + \frac{4}{\gamma^4}}$$



Why? An eigenvalue problem



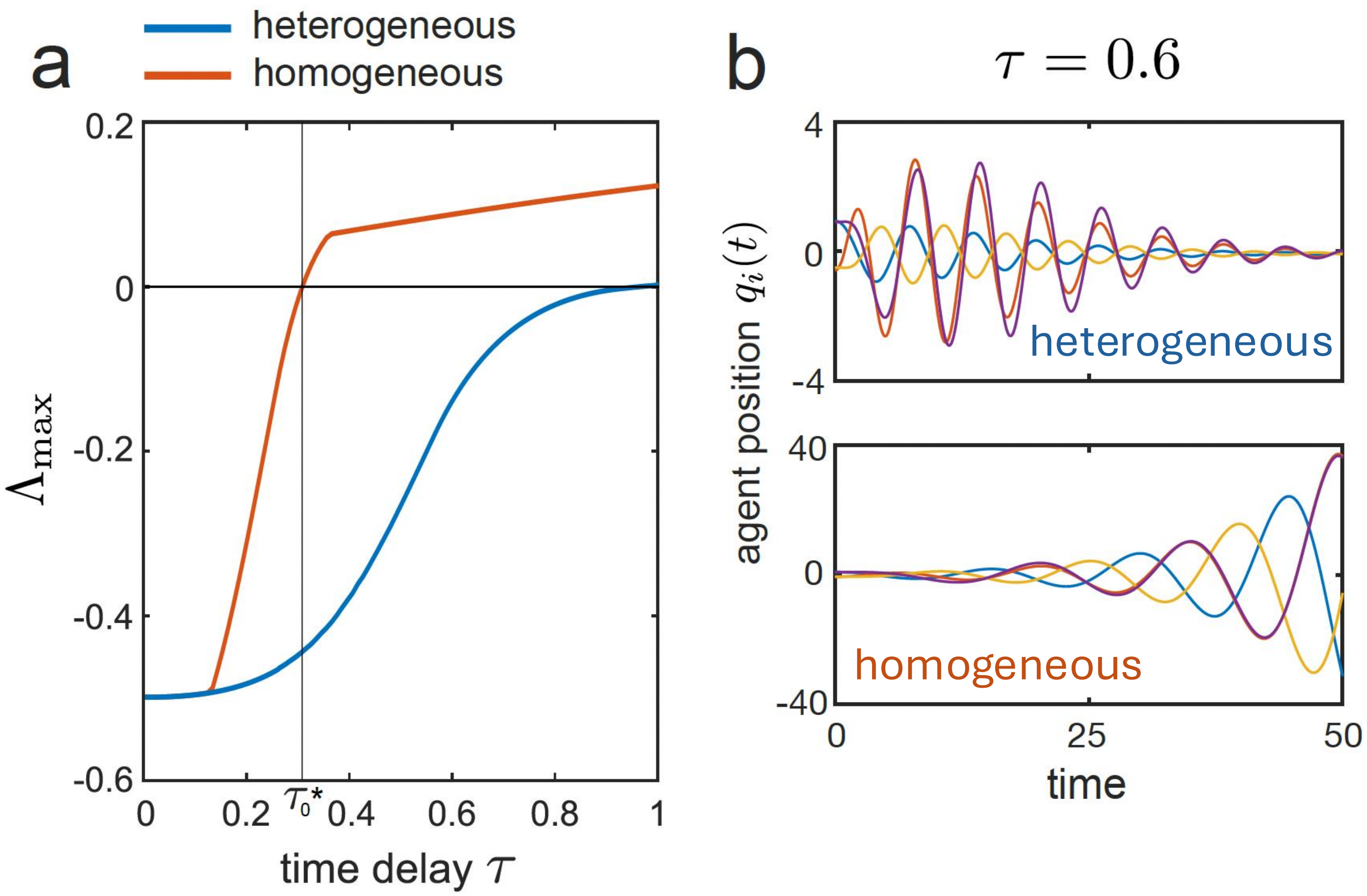
Questions

Which conditions lead to a heterogeneous optimum?

How to efficiently locate them (algorithmically)?

What are the implications for other network dynamics?

Time-delay multi-agent systems

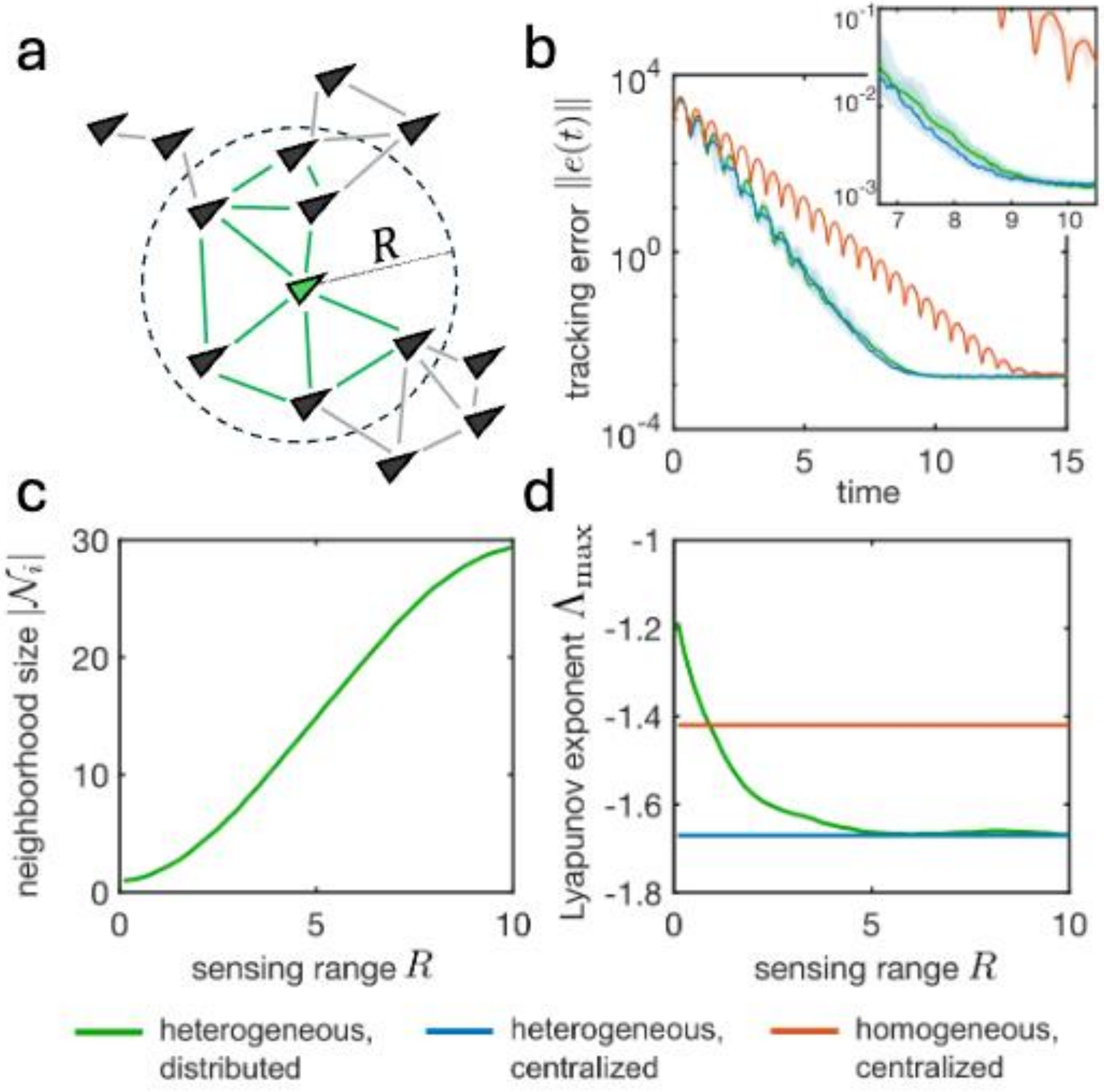


time-delay multi-agent model

$$\dot{\boldsymbol{q}}_i(t) = \boldsymbol{p}_i(t),$$

$$\dot{\boldsymbol{p}}_i(t) = -k_i \left(\sum_{j=1}^{N} L_{ij} \boldsymbol{q}_j(t-\tau) + \sum_{j=1}^{N} L_{ij} \boldsymbol{p}_i(t-\tau) \right)$$

Distributed eigenvalue optimization



Special thanks

Center for Network Dynamics

Optimal flock formation induced by heterogeneity

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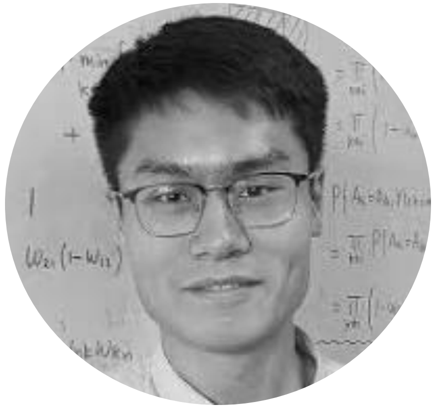
arXiv:2504.12297

Ana Barioni

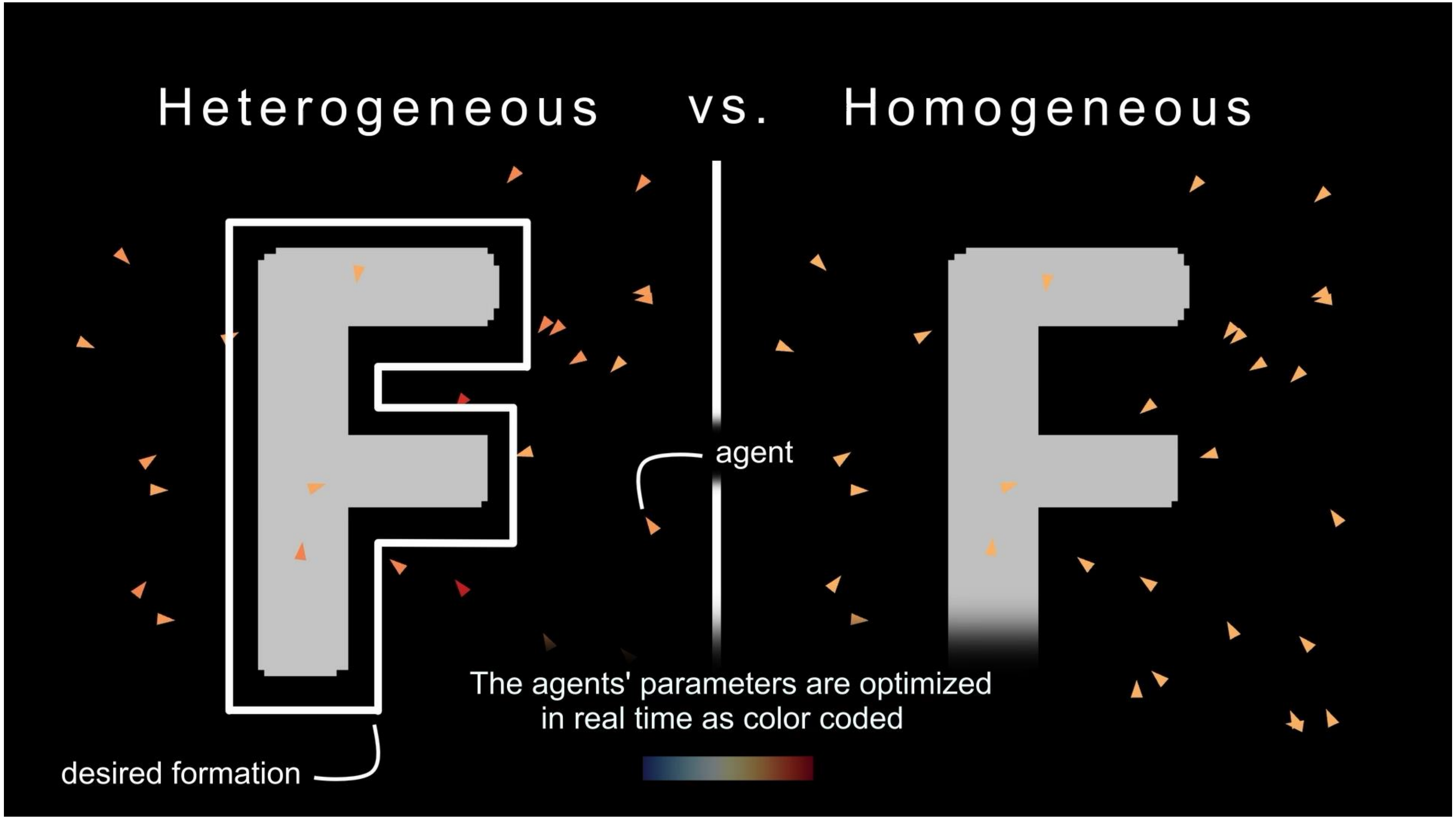


Chao Duan

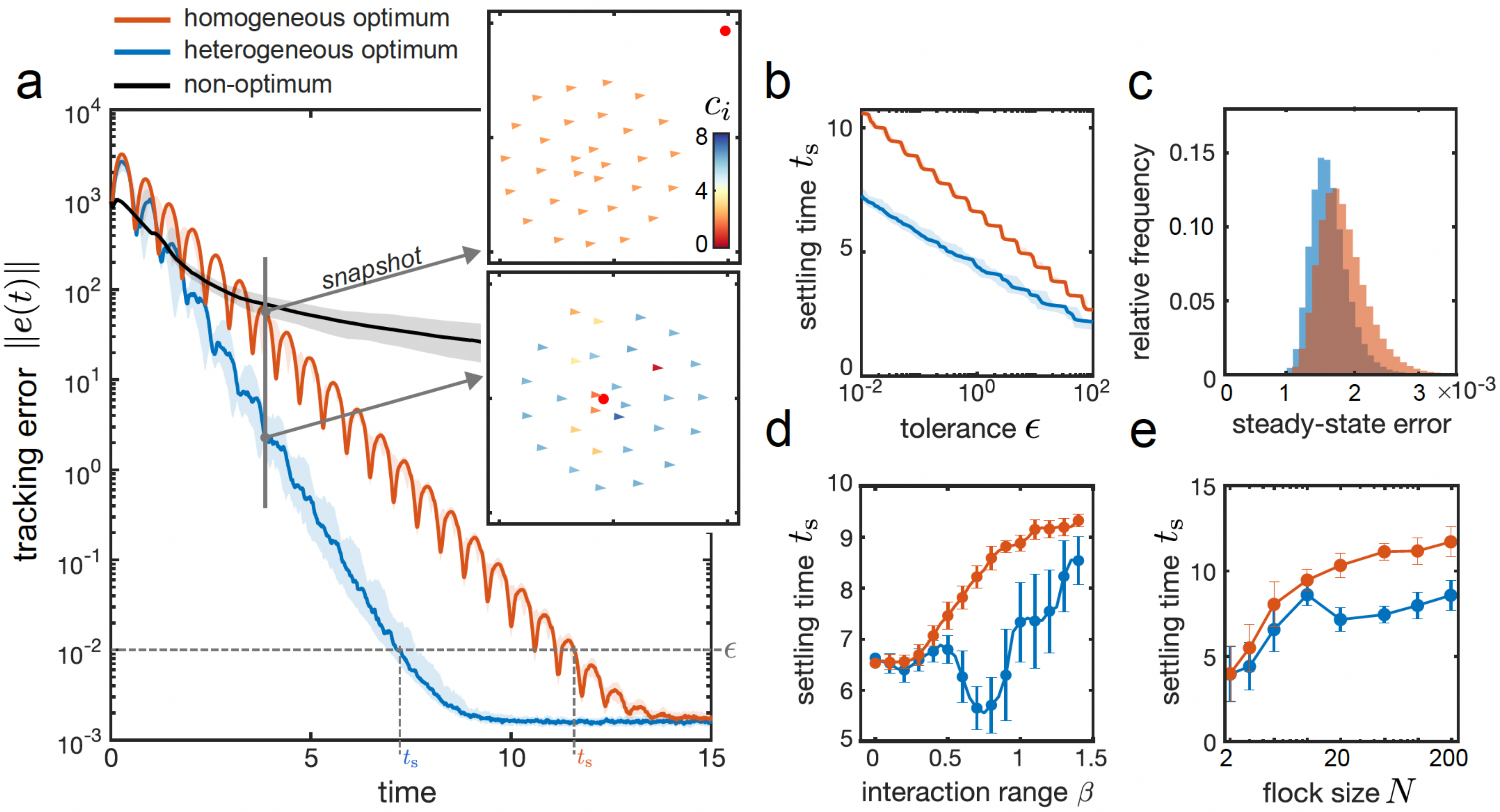
Adilson Motter

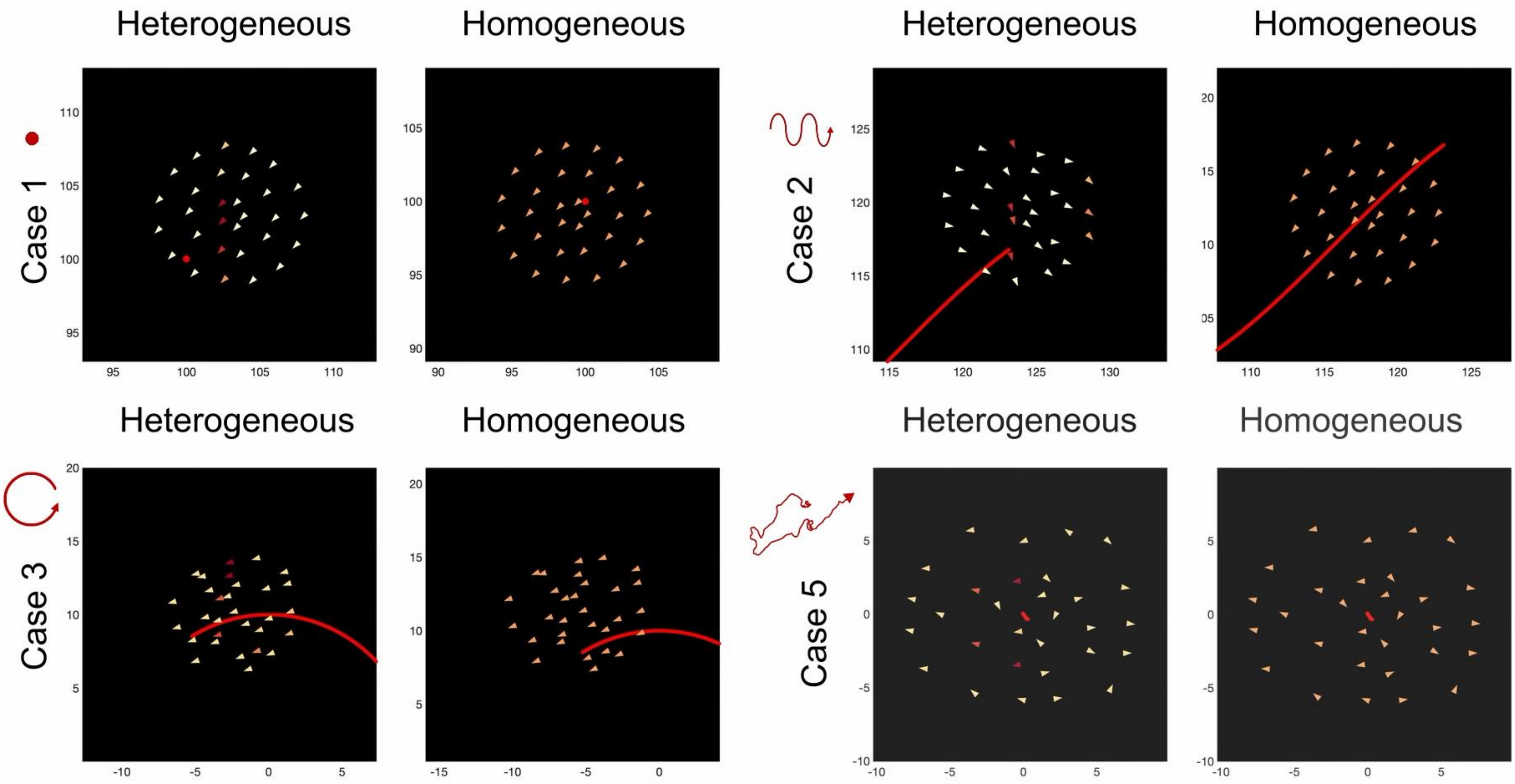
Camila Felix

Thank you! arXiv:2504.12297 ... Questions?



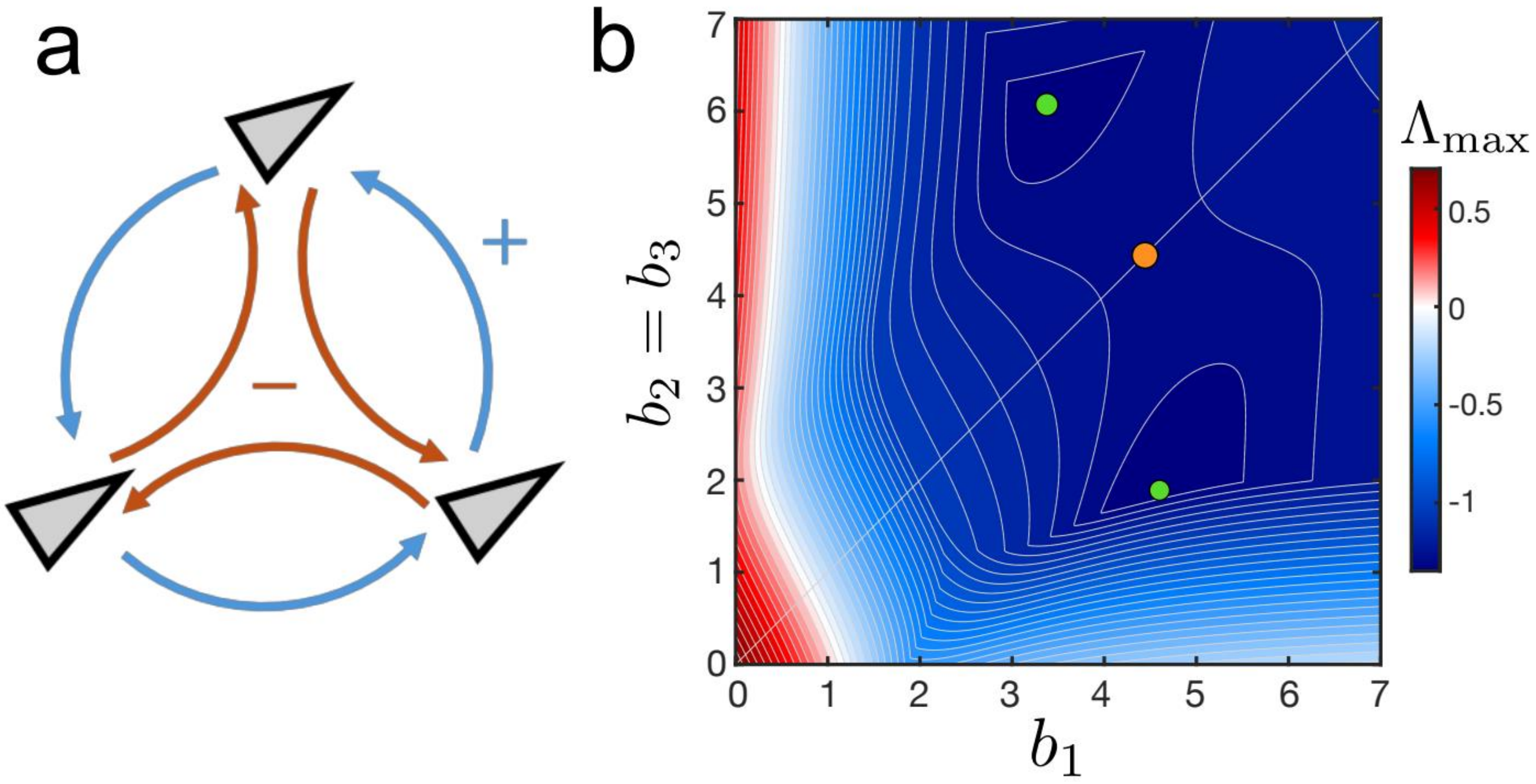
Heterogeneous vs homogeneous flocking



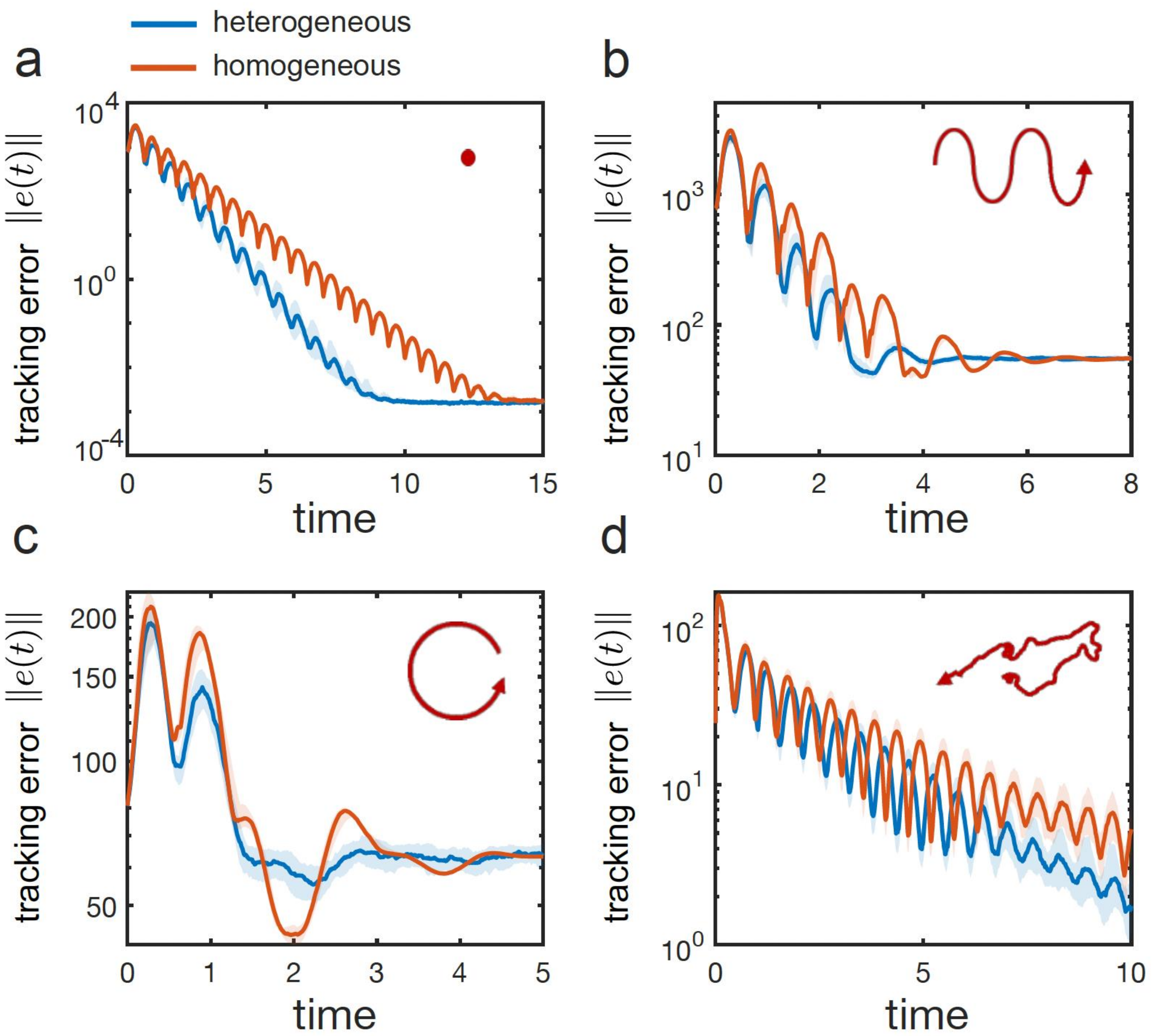


Heterogeneous
Homogeneous
Heterogeneous
Homogeneous
Case 1
Case 2
Heterogeneous
Homogeneous
Heterogeneous
Homogeneous
Case 3
Case 5

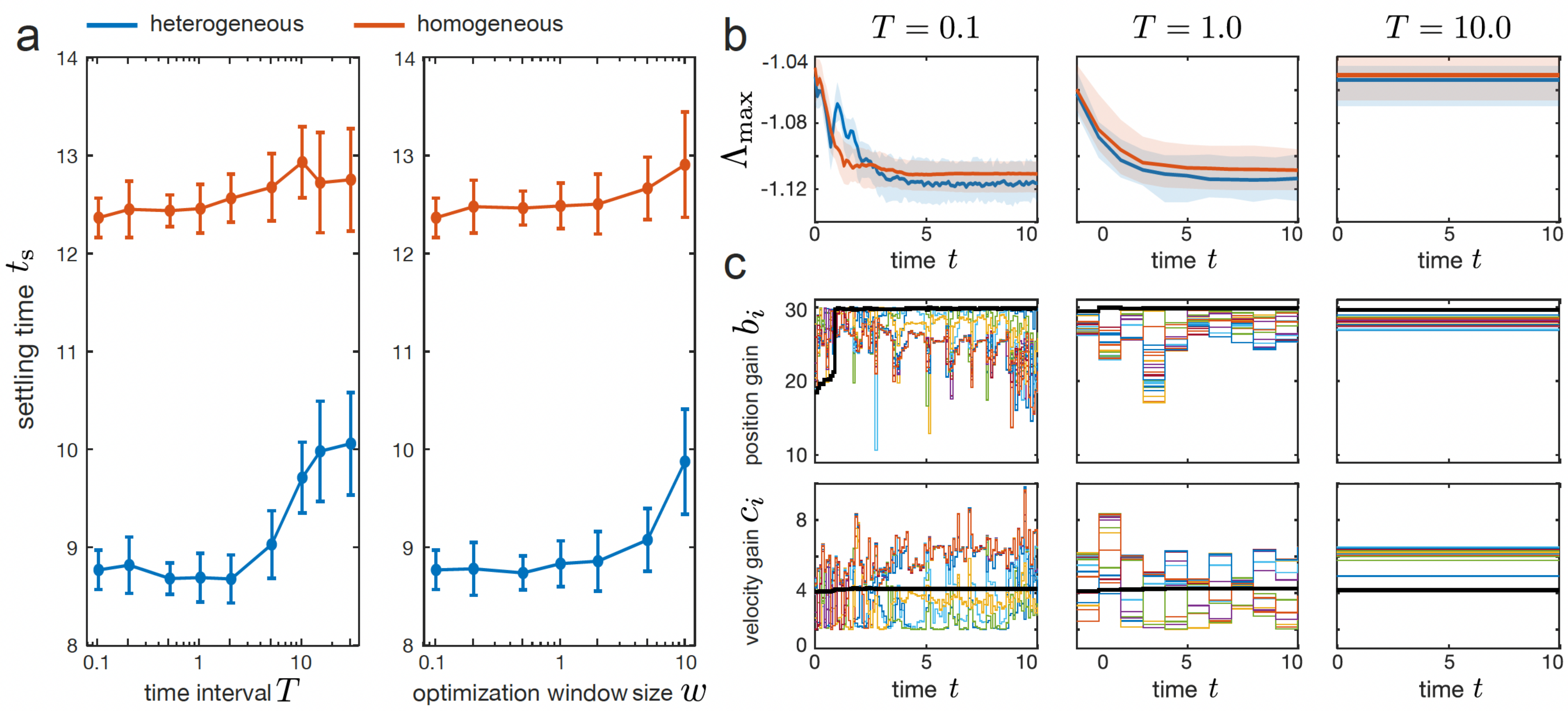
Why? An eigenvalue problem



Heterogeneous vs homogeneous flocking



Heterogeneous vs homogeneous flocking



Yet another application (setpoint tracking)